REPORT ONR-CR215-230-3F

OFFICE OF NAVAL RESEARCH

A LIFTING SURFACE THEORY FOR WINGS EXPERIENCING LEADING-EDGE SEPARATION

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CONTRACT N00014-75-C-0257
ONR TASK 215-230

30 JUNE 1977

FINAL REPORT 1 JANUARY 1975 - 30 JUNE 1977

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: ONR CR215-230-3F	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): A Lifting Surface Theory for Wings Experiencing Leading-Edge Separation.	5. TYPE OF REPORT & PERIOD COVERED: Final Report, 1 January 1975 — 30 June 1977
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Thomas K. Matoi and Sheila E. Widnall	8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0257
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61153N-14, RR014-11-84, NR215-230
11. CONTROLLING OFFICE NAME AND ADDRESS: Department of the Navy, Office of Naval Research, Vehicle and Weapons Technology Division, Vehicle Technology Program Code 211, Arlington, VA 22217	12. REPORT DATE: June 1977
	13. NUMBER OF PAGES: 145
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Office of Naval Research Resident Representative, E19-628, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; Distribution unlimited

146p. RR01411 RR0141184

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Leading-edge vortex
Lifting surface theory
Vortex flow phenomena

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes a nonlinear lifting surface theory for a wing with leading-edge vortices in a steady, incompressible flow. A numerical scheme has been developed from this theory and initial runs have been made for the delta wing and arrow wing planforms. A general procedure for other planforms is also described. The present formulation is the result of an extensive modification of the work of Nangia and Hancock, in which a model of the leading-edge vortex is added to a vorticity representation

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S/N 0102-014-6601

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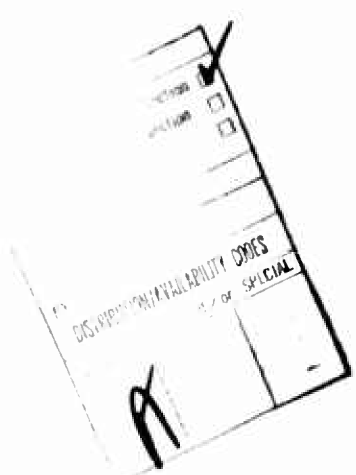
of the wing and wake. This lifting surface theory program is based on the kernel function formulation, in that the vorticity distribution is described by continuous functions with unknown coefficients. The vortex location is similarly described by functions with unknown coefficients. These unknowns are found by satisfying the downwash condition and the no-force condition on the leading-edge vortex representation. Due to the nonlinear nautre of the boundary conditions with respect to the vortex position, the solution is obtained from an iterative scheme based on Newton's method. Results for the delta wing and arrow wing are presented and compared with experiment and other theories. These results indicate that reasonable predictions can be obtained although the computational effort is considerable. Finally, areas of future investigations suggested by the present work are given.



SUMMARY

This report describes a nonlinear lifting surface theory for a wing with leading-edge vortices in a steady, incompressible flow. A numerical scheme has been developed from this theory and initial runs have been made for the delta wing and arrow wing planforms. A general procedure for other planforms is also described. The present formulation is the result of an extensive modification of the work of Nangia and Hancock, in which a model of the leading-edge vortex is added to a vorticity representation of the wing and wake. This lifting surface theory program is based on the kernel function formulation, in that the vorticity distribution is described by continuous functions with unknown coefficients. The vortex location is similarly described by functions with unknown coefficients. These unknowns are found by satisfying the downwash condition and the no-force condition on the leading-edge vortex representation. Due to the nonlinear nature of the boundary conditions with respect to the vortex position, the solution is obtained from an iterative scheme based on Newton's method. Results for the delta wing and arrow wing are presented and compared with experiment and other theories. These results indicate that reasonable predictions can be obtained although the computational effort is considerable. Finally, areas of future investigations suggested by the present work are given.

ACKNOWLEDGMENTS

The authors would like to thank Professors Eugene Covert and Mårten Landahl for their comments and suggestions during the period of research.

The investigation described in this report was supported by the Office of Naval Research, Code 211, under Contract N00014-75-C-0257, under the supervision of Dr. Robert Whitehead of the Vehicle Technology Division.

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1. Introduction

Supersonic aircraft generally employ highly swept wings with thin leading edges in an effort to reduce drag in their operational environment. This wing design results in leading-edge separation at even low angles of attack, typically about 5°.

Although theoretical predictions are generally excellent for unseparated flow outside the transonic range, the vortex-wing interaction problem has been successfully attacked only recently for general planforms. The difficulty introduced by the separation is two-fold. First, the location of the separated vorticity in a theoretical model is not known a priori. Secondly, due to the large spanwise velocities induced by the presence of the vortex on the wing, the pressure calculations must include non-linear terms as well as the classical linear contribution. Due to the non-linear nature of the boundary condition which is needed to determine the location of the separated vorticity, an iterative procedure must be used to determine the flow field. Details of early efforts to describe, measure, and predict the effects of flow separation are chronicled in Matoi (1975)[1], Smith (1975)[2], and elsewhere.

The leading-edge separation phenomena has been documented for many planforms, but the delta wing has received the greatest share of attention, due to its inherent simplicity. A description of the flow about a delta wing was given by Örnberg (1954)[3], and one of his illustrations is presented in Figure 1, where the separated vortex sheet is seen to feed a primary vortex core, which then induces a secondary separation from the upper surface of the wing.

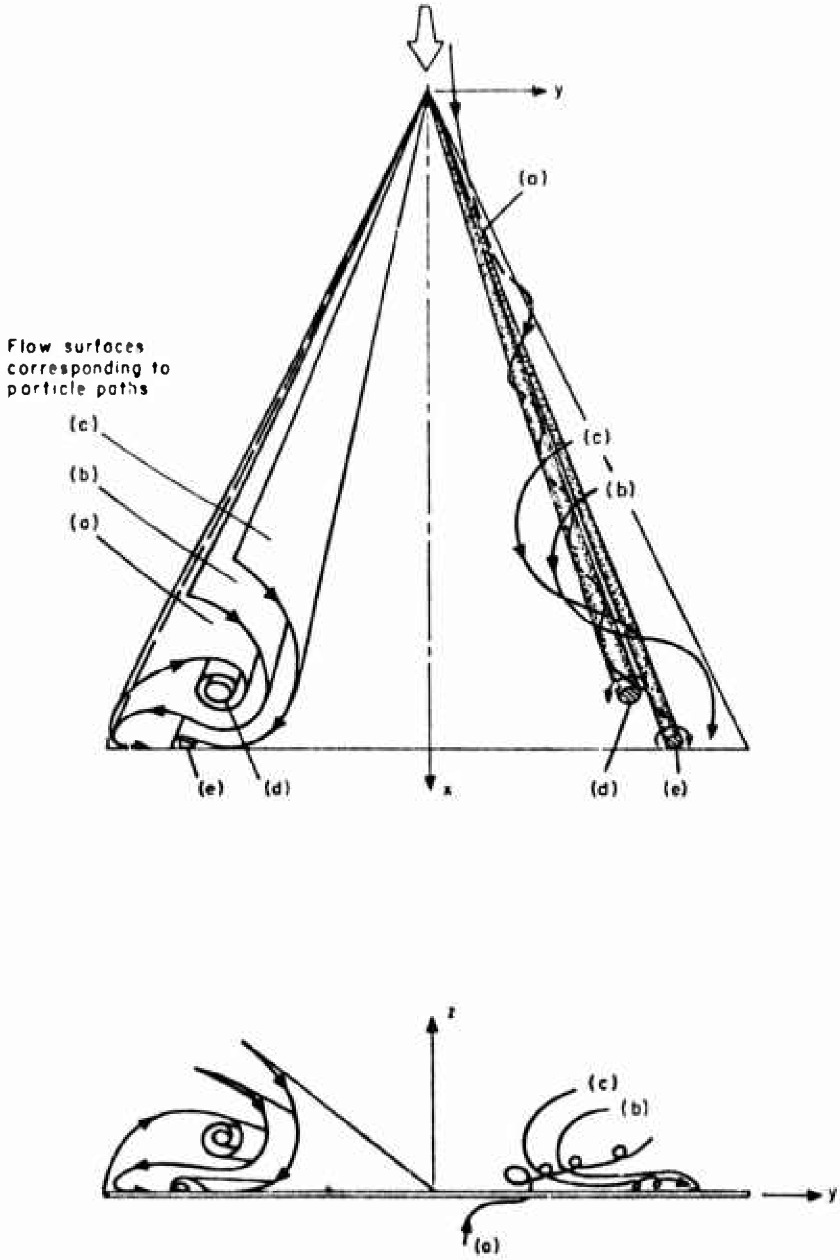


Figure 1. Schematic sketches showing flow on suction side of 70° flat plate delta wing at $\alpha=15^{\circ}$ [after Örnberg(1954)].

This secondary vortex results from the separation of the viscous boundary layer on the wing, when it encounters the adverse pressure gradient present on the upper surface. Since this line of separation can only be located by a viscous analysis, this additional complexity has been ignored in the following models.

An early effort to theoretically predict this flow field was made by Brown and Michael (1955)[4]. They considered a conical, flat-plate delta wing at moderate angles of attack under the additional restriction of slender-body theory. They modeled the vortex core by a line vortex whose strength increased linearly along its axis. The vortex was fed by a cut, i.e., a feeding sheet from the leading edge which was restricted to the cross-flow plane. This model of the vortex sheet will be referred to as a vortex-cut model.

Smith (1966)[5] refined the Brown and Michael model to include a representation of the actual force-free vortex sheet as well as the vortex core. In Figure 2 (top) the vortex-sheet and vortex-core location are presented in the cross-flow plane for various extents of the vortex sheet. α designates the angel of attack and λ is the leading-edge sweep angle. The extent of the sheet obviously increases as one increases the fraction (F) of the total shed vorticity which is included in the sheet. These results were obtained by running an amended version of the program provided by Pullin (1973)[6]. Pullin used a representation of the leading-edge vortex sheet similar to the one employed by Smith, but developed a more systematic iteration procedure for finding the stable configuration of the shed vorticity. The case of no sheet (F = 0) corresponds to the Brown and Michael model. Increasing the extent of the sheet beyond F = .19 results in little change for the parameters

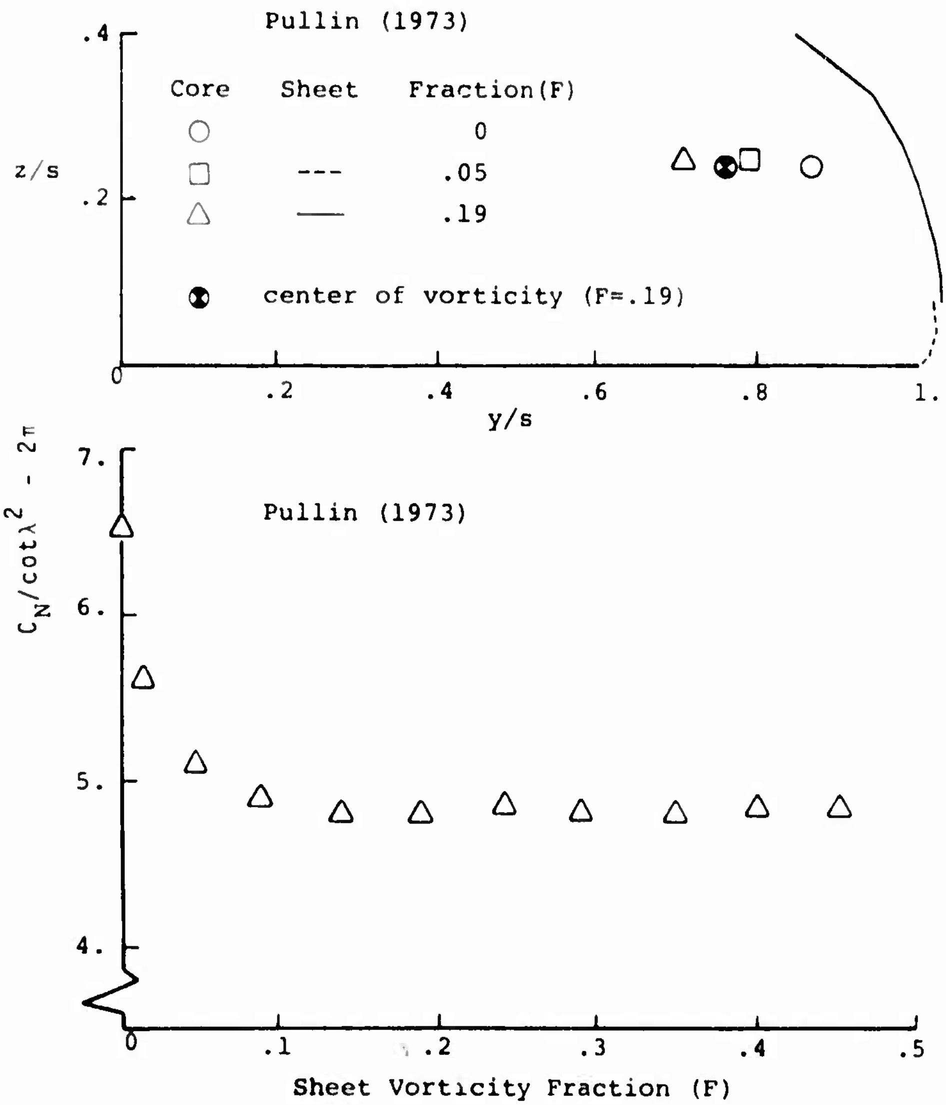


Figure 2. Dependence of vortex-sheet shape and core location (top) and convergence of nonlinear part of normal force (bottom) on the fraction of total shed vorticity contained in the sheet for a delta wing ($\sin\alpha/\cot\lambda = 1$) from slender-body theory.

considered. As can be seen, the effect of introducing the vortex sheet is to move the vortex core inward. It must be noted, that the center of shed vorticity no longer corresponds to the location of the core, and for the case plotted ($F = .19$), the center of vorticity is located at $y/s = .76$, $z/s = .24$. In the lower part of Figure 2, the convergence of the non-linear normal force contribution is presented. This indicates that global quantities may be obtained by considering only a small segment of the vortex sheet.

Recently, the restriction of slender-body theory has been removed, and the general three-dimensional separated problem has now been considered. Lifting-surface theories have been developed along two distinct lines. First, there are the finite-element methods where the wing is replaced by a number of discrete vortex elements and their strengths are determined by satisfying the appropriate boundary conditions. The leading-edge vortex problem has been attacked by finite-element methods by Kandil, Mook, and Nayfeh (1974)[7] and Brune, Weber, Johnson, Lu, and Rubbert (1975)[8].

The alternate method is to represent the vorticity distribution on the planform by a set of loading functions whose coefficients are chosen to satisfy the boundary conditions. This method is called the kernel-function method. In Matoi, Covert, and Widnall (1975)[9], a lifting-surface theory for separated flow based on the kernel-function method was developed for a delta wing. The purpose of this report is to improve and extend the development of that kernel-function procedure.

2. Problem Formulation

The reasons for choosing the kernel-function method over the finite-element method have been detailed in the earlier report by Matoi, et al. (1975)[9]. It was believed that such a procedure could be more easily generalized to include unsteady effects, vortex-breakdown models, and other extensions, and would alleviate some of the difficulties encountered when using discontinuous finite-element procedures. These difficulties, which include "lost" vortices in the line-vortex models and convergence problems as the number of elements is increased, result from the infinite discontinuity in the velocity between discrete panels or at the vortex element. Artifices (such as the introduction of viscosity, finite core radius, or other smoothing procedures) are needed to alleviate this feature of the discrete vortex models. The only other work employing continuous loading functions found in an extensive literature search was by Nangia and Hancock (1968)[10]. Many of the symbols and much of the present formulation have their origin in that report.

The coordinate system used in this report is presented in Figure 3. The planform is presently considered to be in the x-y plane. The non-planar problem can also be considered if a spanwise coordinate is used instead of y. The planform can be completely general. The configuration is considered to be symmetric, and the flow field can be described by satisfying the boundary conditions on the right side alone. The boundary conditions on the other side are automatically satisfied by symmetry. However, the asymmetric problem (e.g., the wing at a sideslip angle) can be considered with minor modifications. See Figure 4 for the representation of the wing, leading-edge vortices and wake. It is to be noted that the vortex-cut model

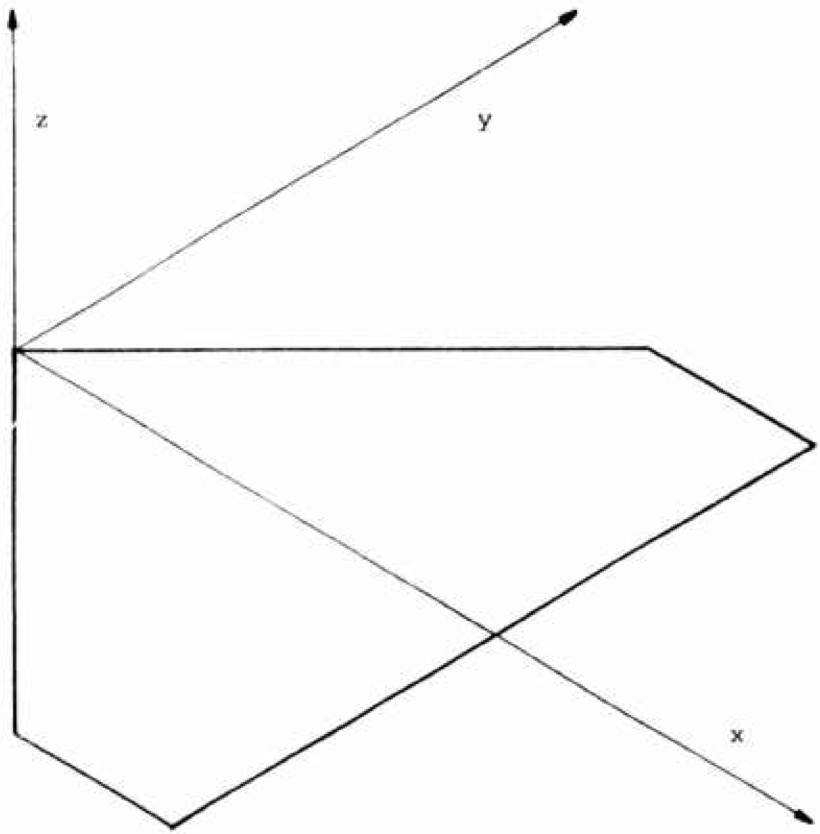


Figure 3. Coordinate system.

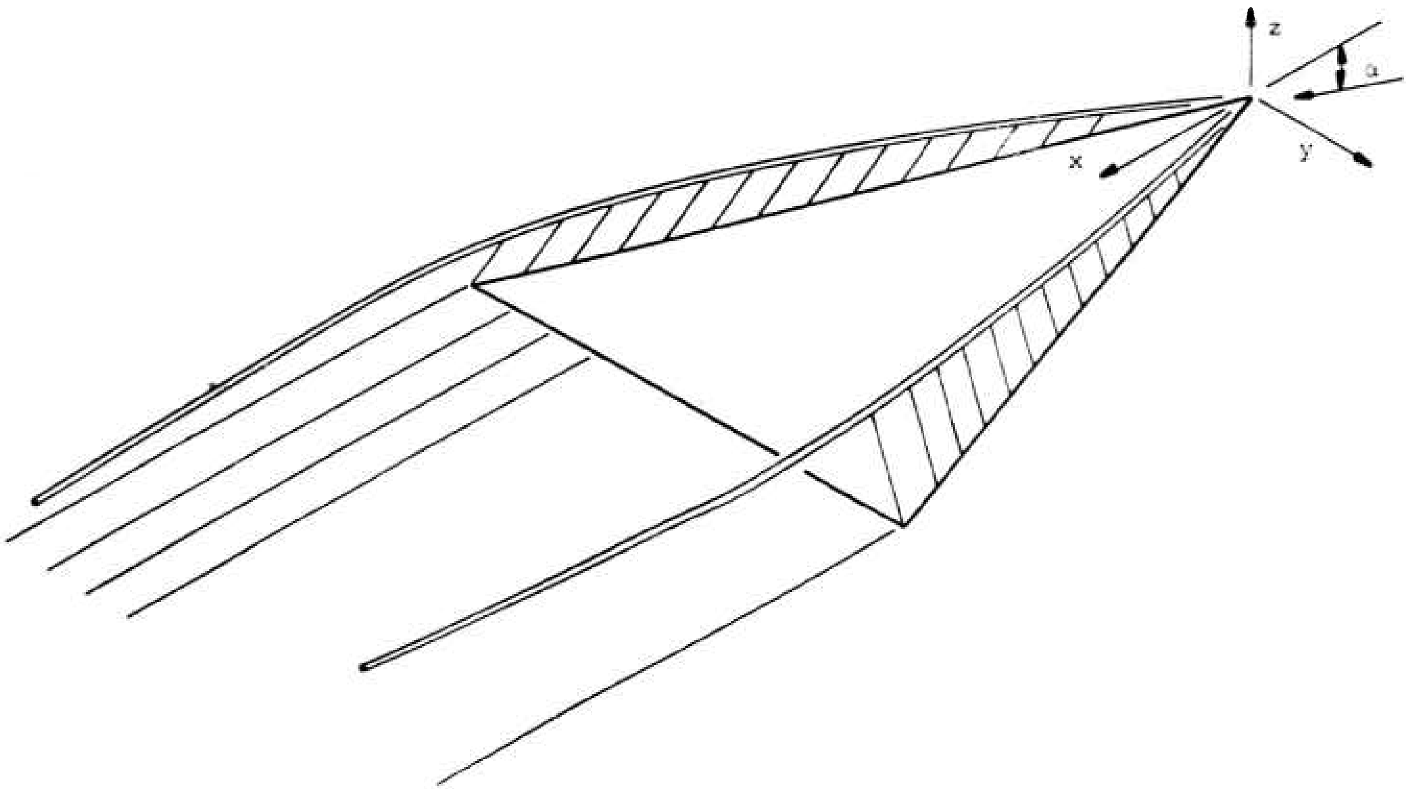


Figure 4. Representation of wing, wake and leading-edge vortices.

of Brown and Michael is presently being used for the reason of simplicity. Later, the more correct representation with some part of the sheet may be included in this type of analysis.

The governing equation in three-dimensional, inviscid, irrotational, steady flow about a wing-body combination is Laplace's equation. The solution can be formulated as an integral equation over the boundary of the aircraft configuration and the regions of shed vorticity. There are several equivalent formulations for the solution, but vortex sheets are used to represent the wing and wake in this report. The velocity distribution can then be given in the following vector form

$$\vec{v}(\vec{r}) = -\frac{1}{4\pi}\int_{S'} \frac{\vec{\gamma}\times(\vec{r}\,' - \vec{r})}{|\vec{r}\,' - \vec{r}|^3}\, dS' \tag{1}$$

where

$$\vec{r}\,' - \vec{r} = (x' - x)\hat{i} + (y' - y)\hat{j} + (z' - z)\hat{k}$$

$$\vec{\gamma} = \gamma_x\hat{i} + \gamma_y\hat{j} + \gamma_z\hat{k}$$

$$\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$$

S' is the surface of integration, $\vec{\gamma}$ is the vorticity vector, $\vec{v}$ is the perturbation velocity vector, i.e., the velocity minus the uniform free stream; and $\vec{r}$ is the radius vector from the origin. The velocities are nondimensionalized with respect to the free stream, and the distances are nondimensionalized with respect to the maximum chordwise length.

Since the vorticity lies in the plane of the wing and wake, the vorticity representing the wing consists of only two non-zero components γ_x and γ_y. Conservation of vorticity can then be written as

$$\frac{\partial \gamma_y}{\partial y} = - \frac{\partial \gamma_x}{\partial x} \tag{2}$$

In the present formulation, the vorticity in the wing and wake is divided into two parts. First, there is a portion -- represented by subscript 1 -- which behaves like the traditional bound vorticity and only leaves the wing at the trailing edge. Secondly, there is a portion -- represented by subscript 2 -- which feeds the leading-edge vortices.

$$\gamma_y \equiv \gamma = \gamma_1 + \gamma_2$$

$$\gamma_x \equiv \delta = \delta_1 + \delta_2 \tag{3}$$

The contributions are chosen so that γ_1 and δ_1 fall to zero at the leading edge, while γ_2 and δ_2 are related so that the vorticity is perpendicular to the leading edge. This is necessitated by the Brown and Michael model employing a vortex-cut combination to insure finite velocities at the leading edge. A more complete model employing a leading-edge vortex sheet representation would utilize a general separation angle which would be fixed by the no-load (Kutta) condition at the leading edge.

The functional forms of the wing vorticity are

$$\gamma_1(\theta,\eta) = \frac{8\pi s}{c(\eta)} \sqrt{1-\eta^2} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{4\, a_{n,m}}{2^{2n}} \; U_{2(m-1)}(\eta) \sin n\theta$$

$$\gamma_2(x,y) = \frac{-y}{\sqrt{x^2 + y^2}} \; \frac{\pi}{2} \sum_{q=1}^{Q} g_q \, (2q - 1) \; \cos \left[\frac{2q-1}{2} \Big/ \sqrt{\frac{x^2 + y^2}{1 + s^2}} \right] \tag{4}$$

where

$$x = 1/2\ [x_{TE}\ (y) + x_{LE}\ (y)\ ] - 1/2\ c(y)\cos\theta$$

$$y = s\eta \tag{5}$$

The distribution for γ_1 was obtained from linearized lifting surface theory and vanishes at the leading and trailing edges. For additional details, see Ashley and Landahl (1965)[11]. U_m are Chebyshev polynomials of the second kind, x_{LE} and x_{TE} are the location of the leading and trailing edges of the planform, respectively, c is the local chord, and s is the semispan. The form of γ_2 insures that the vorticity feeding the leading-edge vortex will be perpendicular to leading edges, which are formed by rays from the apex. Modes similar to these were developed by Nangia and Hancock (1968)[10] for the three-dimensional delta wing. The coefficients $a_{n,m}$ and g_q are the unknown coefficients of the vorticity functions. The leading-edge vortex strength is defined by

$$\Gamma(x) = \sum_{q=1}^{Q} g_q \sin\left[\ (2q-1)\pi x/2\right] \tag{6}$$

where the modes have been chosen such that $\frac{d\Gamma}{dx} = 0$ at the trailing edge, i.e., there is no additional feeding of wing vorticity into the leading-edge vortex at the trailing edge. See Figure 5 for a representation of the bound vorticity component γ_2. Using Equation 2, one can obtain

$$\delta_1(\theta,\eta) = -\frac{4\pi}{\sqrt{1-\eta^2}} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{4\ a_{n,m}}{2^{2n}} \left[1/2\ \left\{-\left[\ (2m-1)\eta + \frac{(1-\eta^2)}{c(\eta)}\ \frac{dc}{d\eta}\ \right] U_{2(m-1)}(\eta) + \right.\right.$$

$$(2m+1)\ U_{2m-3}\ (\eta)\} * [\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1}]$$

$$- \frac{2n(1-\eta^2)}{c(\eta)} \quad U_{2(m-1)} \left\{ \left(\frac{dx_{LE}}{d\eta} + 1/2 \frac{dc}{d\eta} \right) \frac{\sin n\theta}{n} \right.$$

$$\left. - 1/4 \frac{dc}{d\eta} \left[\frac{\sin (n-1)\theta}{n-1} + \frac{\sin (n+1)\theta}{n+1} \right] \right\} \Bigg\} \qquad (7)$$

where

$$U_{-1}(\eta) \equiv 0$$

The wake is assumed to be flat and to possess the trailing vorticity distribution imparted at the trailing edge, as in linear lifting surface theory. Consequently, the trailing wake adds no new parameters.

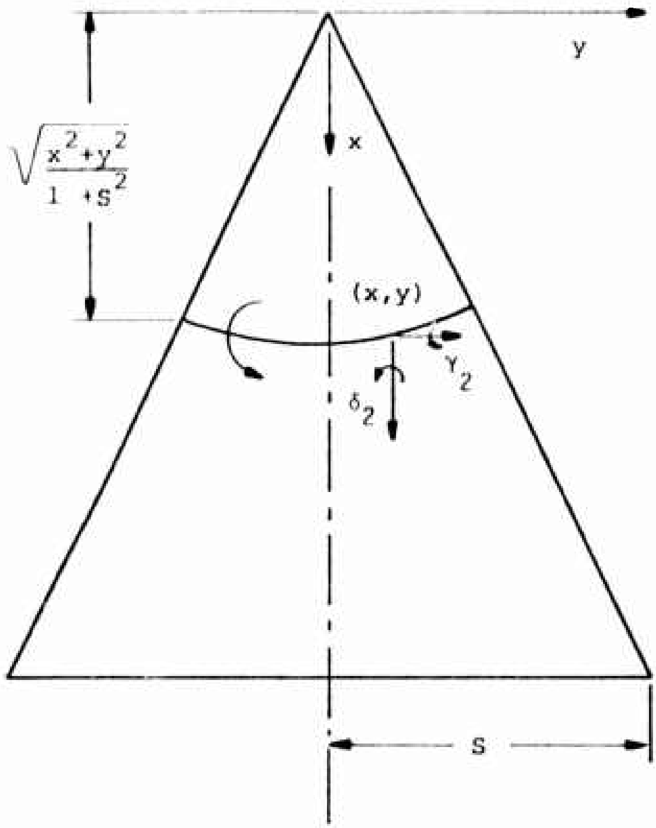
Finally, the location of the leading-edge vortex is defined by the polynomials

$$y_v(x) = \sum_{\ell=1}^{L} g_{y_v}\ T_{2\ell-1}(x)$$

$$z_v(x) = \sum_{\ell=1}^{L} g_{z_v}\ T_{2\ell-1}(x) \qquad (8)$$

where T_ℓ are Chebyshev polynomials of the first kind. This introduces the final set of unknowns, g_{y_v} and g_{z_v}.

After the mode shapes have been defined, it is necessary to satisfy the appropriate boundary conditions to determine the unknown coefficients. The applicable boundary conditions are the no-flow condition through the wing and the no-load condition on the free vortex sheet and on the leading-edge vortex-cut combination.



$$\gamma_2(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \frac{d\Gamma(x_e)}{dx} \qquad x_e = \sqrt{\frac{x^2+y^2}{1+s^2}}$$

$$\delta_2(x,y) = \frac{y}{\sqrt{x^2+y^2}} \quad \frac{d\Gamma(x_e)}{dx} \qquad x_e = \sqrt{\frac{x^2+y^2}{1+s^2}}$$

Figure 5. Representation of bound vorticity feeding leading-edge vortex.

The downwash condition becomes

$$w = -\sin\alpha \tag{9}$$

on the wing (z = 0), where α is the angle of attack and w is the upwash induced by the vorticity distribution. This requires the evaluation of the w component of the integral in Equation 1 at a set of collocation points. The cosine distribution of Hsu (1957)[12], modified for separated flow, is given in Figure 6 for five chordwise station (NCORD = 5) by five spanwise stations (NSPAN = 5). A large percentage of the computation time is presently consumed by the calculation of the contribution from the wing surface, since the denominator contains a singularity at the collocation points.

A further distinction must now be made between the contributions in Equation 9. Its various components are distinguished by the nature of their contribution to the vertical velocity.

First, since the bound vorticity related to γ_1, described in Equation 4, only leaves at the trailing edge, horseshoe vortices are used to represent this contribution, which corresponds to an integration in the x direction of Equation 1. Thus, for that contribution, the relevant vorticity component becomes

$$w_1\,(x,y,z) = \frac{1}{4\pi} \int_{-s}^{s} \int_{x_{LE}}^{x_{TE}} \gamma_1 \; Kw \; dx'dy' \tag{10}$$

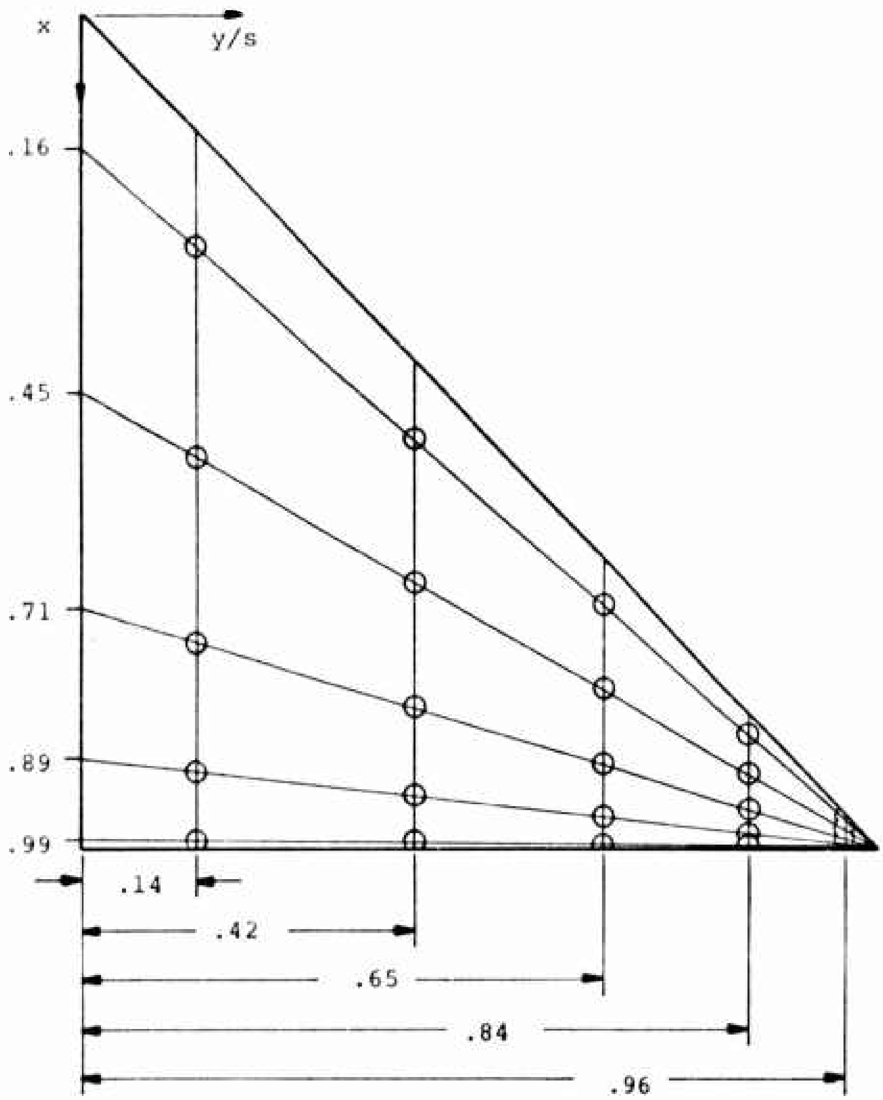


Figure 6. Collocation points for downwash on right half of wing (NSPAN = 5, NCORD = 5).

where

$$Kw \equiv \frac{1}{(y-y')^2 + z^2} \left\{ \left[1 + \frac{x - x'}{\sqrt{(x-x')^2+(y-y')^2 + z^2}}\right] \cdot \left[1 - \frac{2z^2}{(y-y')^2+z^2}\right] - \frac{z^2 \, (x-x')}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} \right\} \tag{11}$$

Thus the singularity for the contribution, γ_1, is limited to the spanwise direction, when $z = 0$, and the contribution from this term to Equation 8 may be readily evaluated. The four integration regions employed for this surface integration are presented in Figure 7 (top). The actual evaluation was performed using a simplified version of a routine available at M.I.T., which was developed by Widnall (1964)[13] for the more general problem of the unsteady, incompressible, non-planar wing. This program was based on an extension of the work by Watkins, Runyan, and Woolston (1959)[14]. Alternate forms, such as the procedure developed by Hsu (1957)[12] could be used instead.

For the contribution from γ_2 and δ_2, the evaluation of the integral in Equation 1 is handled in two parts. The integral over the wing surface Sw can be rewritten as

$$w_2(x,y,z) = \frac{1}{4\pi} \iint_{Sw} \frac{(x'-x)\,\gamma_2(x',y') - (y'-y)\,\delta_2(x',y')}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} \, dx'dy' \tag{12}$$

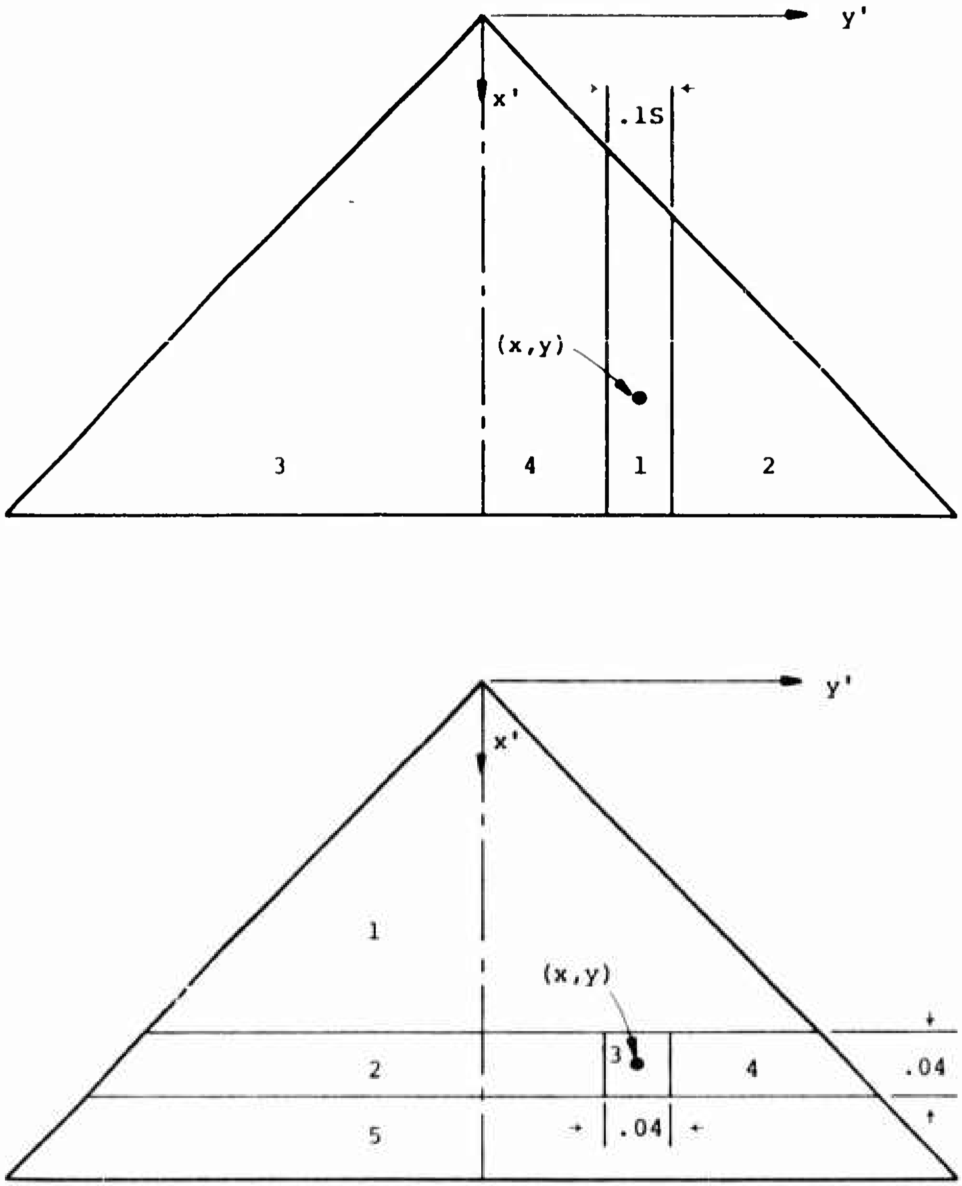


Figure 7. Regions of integration for calculating upwash coefficients at the point (x,y) for the γ_1 contribution (top) and for the γ_2, δ_2 contribution (bottom).

On the wing surface, z = 0, the integral may be rewritten to isolate the singularity.

$$w_2(x,y,o) = \frac{1}{4\pi} \int\int_{Sw} dS \frac{(x'-x)[\gamma_2(x',y') - \gamma_2(x,y)] - (y'-y)[\delta_2(x',y') - \delta_2(x,y)]}{[(x - x')^2 + (y - y')^2]^{3/2}}$$

$$+ \frac{1}{4\pi} \gamma_2(x,y) \oint\oint_w \frac{(x'-x)\, dx'dy'}{[(x-x')^2 + (y-y')^2]^{3/2}}$$

$$+ \frac{1}{4\pi} \delta_2(x,y) \oint\oint_w \frac{(y-y')\, dx'dy'}{[(x-x')^2 + (y-y')^2]^{3/2}} \tag{13}$$

The first term can now be evaluated numerically, while the remaining two terms are evaluated in Appendix A. Figure 7 (bottom) gives the five integration regions employed for this surface integration for the delta wing. Each region is covered by a 24 x 24-point Gaussian quadrature. Fortunately, this computation does not require any iteration and is performed once for a given set of collocation points and wing planform. Since this calculation and a related integral in the no-force condition consume much of the computational effort, significant reductions in this integration would be advantageous.

Additional contributions to the downwash on the wing are obtained from the wake aft of the wing and from the leading-edge vortices.

As described previously in Figure 4, the spanwise component of the vorticity, γ_2, is assumed to be zero aft of the trailing edge, while the streamwise component, δ_2, is only a function of the spanwise variable in the wake. Thus, one obtains the following contribution from the wake according to Equation 1.

$$w_T(x,y,z) = -\frac{1}{4\pi}\iint_{S_T} \frac{(y'-y)\,\delta_2(y')\,dy'dx'}{[(x'-x)^2 + (y'-y)^2 + z^2]^{3/2}} \tag{14}$$

where S_T represents the wake surface. The streamwise integral may be performed explicitly for a given planform to yield

$$W_T(x,y,z) = -\frac{1}{4\pi}\int_{-s}^{s} (y'-y)\,\delta_2(y')\,I(y';\,x,y,z)\,dy' \tag{15}$$

where

$$I\,(y';x,y,z) = \frac{1}{(y'-y)^2 + z^2}\left[1 - \frac{x_{TE}(y') - x}{\sqrt{(x_{TE}(y') - x)^2 + (y'-y)^2 + z^2}}\right] \tag{16}$$

The function $x_{TE}(y)$ describes the location of the trailing edge of the specified planform. On the wing surface, $z = 0$, this reduces to

$$W_T(x,y,o) = -\frac{1}{4\pi}\int_{-s}^{s} \frac{\delta_2(y')}{y'-y}\left[1 - \frac{x_{TE}(y') - x}{\sqrt{(x_{TE}(y') - x)^2 + (y'-y)^2}}\right] dy' \tag{17}$$

Finally, the contribution from the leading-edge vortices is

$$W_\Gamma(x,y,z) = \int_0^\infty \Gamma(x')\,[f_w(x';x,y,z) + f_w(x';x,-y,z)]\,dx' \tag{18}$$

where

$$f_w(x';\,x,y,z) = \frac{1}{4\pi}\,\frac{(x'-x)\dfrac{dy_v(x')}{dx'} - (y_v(x')-y)}{[\,(x'-x)^2 + (y_v(x')-y)^2 + (z_v(x')-z)^2]^{3/2}} \tag{19}$$

This is the upwash velocity induced by two semi-infinite line vortices according to the Biot-Savart law. The contribution on the wing is obtained by calculating this velocity component at the appropriate control point.

Thus, Equation 9 becomes

$$w = w_1 + w_2 + w_T + w_\Gamma = - \sin\alpha \tag{20}$$

The no-load condition on the trailing vortex sheet and the Kutta condition at the trailing edge of the wing are essential in distinguishing this fully three-dimensional problem from earlier slender-body and conical models. First, the Kutta condition at the trailing edge requires that the vorticity vector be parallel to the velocity vector

$$\frac{\gamma\ (x_{TE}(y),y)}{\delta\ (x_{TE}(y),y)} = \frac{v}{\cos\alpha + u} \tag{21}$$

However, the results of Brune, et al. (1975)[8] indicate that the spanwise vorticity component, γ , is approximately zero at the trailing edge for the delta wing case. This corresponds to the Kutta condition applied in linear lifting surface theory. Thus, the method employed here was to set the spanwise vorticity component, γ , equal to zero aft of the trailing edge as was previously illustrated in Figure 4., i.e., the linear boundary condition has been applied on the trailing vortex sheet rather than the nonlinear one. The form of γ_1 (Equation 4) insures that this component vanishes smoothly at the trailing edge to satisfy the linear Kutta condition there. However, the contribution from γ_2 (Equation 4) does not

automatically vanish at the trailing edge, and consequently, there is a discontinuous transition in this contribution. The use of the linear boundary condition on the wake seems justified, since the major cause of the nonlinearity in the present problem is the presence of the leading-edge vortices which induce high spanwise velocities on the planform. Kandil, et al. (1974, p. 13)[7] noted that "Numerical experiments indicate that the wake adjoining the trailing edge does not exert a strong influence on the results."

Finally, the condition of no-load on the vortex-cut combination is formulated on the right-hand vortex as an extension of the Brown and Michael model. The force components per unit length in the y and z directions are given by F_y and F_z, respectively.

$$F_y/2\Gamma = -\frac{dz_v}{dx} - \frac{1}{\Gamma}\frac{d\Gamma}{dx} z_v + w_i + \sin\alpha$$

$$F_z/2\Gamma = \frac{dy_v}{dx} + \frac{1}{\Gamma}\frac{d\Gamma}{dx}[y_v - y_{LE}(x)] - v_i \tag{22}$$

where w_i and v_i are the velocities induced by the vorticity distribution, excluding the contributions of the right-hand vortex on itself due to curvature. The calculations of the velocity compont w_i at collocation points along the leading-edge vortex requires the evaluation of terms similar to those developed for Equation 20.

Specifically,

$$w_i = w_1 + w_2 + w_T + w_{\Gamma_L} \tag{23}$$

where

$$w_{\Gamma_L}\ (x,y,z) = \int_0^{\infty} \Gamma\ (x')\ fw(x';x,y,z)\ dx' \tag{24}$$

The integrands no longer possess a singularity, and the integral in Equation 12, for example, is performed by two 24 x 24-point Gaussian quadratures.

Meanwhile, the contributions for v_i can be developed in a parallel manner

$$v_i = v_1 + v_2 + v_T + v_{\Gamma_L} \tag{25}$$

where

$$v_1\ (x,y,z) = \frac{1}{4\pi} \int_{-s}^{s} \int_{x_{LE}}^{x_{TE}} \gamma_1\ Kv\ dx'dy' \tag{26}$$

with

$$Kv \equiv \frac{-z(y-y')}{(y-y')^2 + z^2} \left\{ \frac{2}{(y-y')^2+z^2} [1+ \frac{x - x'}{\sqrt{(x-x')^2+(y-y')^2+ z^2}}] + \right.$$

$$\left. [\frac{x - x'}{(x-x')^2 + (y-y')^2 + z^2}]^{3/2} \right\} \tag{27}$$

and

$$v_2(x,y,z) = - \frac{z}{4\pi} \int\int_{S_w} \frac{\delta_2(x',y')\ dx'dy'}{[(x'-x)^2 + (y'-y)^2 + z^2]^{3/2}} \tag{28}$$

$$v_T(x,y,z) = - \frac{z}{4\pi} \int_{-s}^{s} \delta_2(y')\ I\ (y';x,y,z)\ dy' \tag{29}$$

$$v_{\Gamma_L} = - \int_0^{\infty} \Gamma(x')f_v(x';x,-y,z)\ dx' \tag{30}$$

where

$$f_v(x';x,y,z) = \frac{1}{4\pi} \frac{z_v(x') - z - (x'-x)\dfrac{dz_v(x')}{dx'}}{[(x'-x)^2 + (y_v(x')-y)^2 + (z_v(x')-z)^2]^{3/2}} \tag{31}$$

The vorticity distribution has again been assumed to depend only on the spanwise variable in the wake. The function I(y) has been defined in Equation 16.

Thus, the original problem of Laplace's equation with its companion boundary conditions has been formulated as a system of nonlinear equations in terms of the unknown vorticity coefficients, $a_{n,m}$ and g_q, and the unknown vortex location coefficients, g_{yv} and g_{zv}. This has the advantage of transforming a set of integro-differential equations (Equations 1, 9 and 22) into a system of algebraic equations which can be solved on a digital computer.

The primary output parameters of vortex location and vorticity distribution on the wing can then be used to obtain the pressure distribution on the wing. One can obtain the lift and the pitching moment by a simple integration of the pressure loading.

The nonlinear pressure difference on the wing is

$$\Delta C_p \equiv C_{p_u} - C_{p_\ell} = -\ 2\Delta u - \Delta(v^2 + u^2) \tag{32}$$

where the pressure coefficient, C_p, represents the pressure nondimensionalized by the dynamic pressure, the difference symbol, Δ, refers to the difference in the quantity between the upper and lower surfaces, which are represented by the subscripts u and ℓ, respectively. Since the quadratic

term from the chord-wise component of velocity, u^2, is small compared to the spanwise contribution, v^2, that term is ignored and the pressure difference can be rewritten in the following form

$$\Delta C_p = -2\gamma + (v_u + v_\ell)\,\delta \tag{33}$$

This form has been selected as the vorticity components, γ and δ, can be readily evaluated once the vorticity coefficients have been found. The second term on the right-hand side includes a factor which is twice the local mean spanwise velocity. On the wing the only non-zero contribution comes from the leading-edge vortices. Thus,

$$(v_u + v_\ell)_{z=0} = 2\int_0^\infty \Gamma(x')\,[f_v(x';x,y,0) - f_v(x';x,-y,0)]\,dx' \tag{34}$$

This concludes the section on problem formulation. The next section will discuss the actual numerical procedure; and areas of difficulty will be detailed.

3. Numerical Procedure

The procedure to calculate the unknowns is now described. The initial program was written for the delta wing, but it was later generalized to include arrow wings. An iterative scheme to satisfy the system of equations provided by the downwash condition and the no-force condition on the vortex-cut combination must be chosen first. The downwash condition is linear in terms of the vorticity coefficients, while the no-force condition is non-linear in all parameters. Therefore, following the earlier procedure developed by Nangia and Hancock (1968)[10], an attempt was made to satisfy the boundary conditions sequentially.

An initial position for the leading-edge vortex is chosen. For example, for the delta wing, the initial location was obtained from the Brown and Michael model. The number of vorticity modes (M,N, and Q) in Equation 4 must be specified. A set of downwash points greater than or equal to the number of vorticity coefficients must then be chosen. The solution of a set of simultaneous linear equations from Equation 20 then provides a first approximation for the unknown vorticity coefficients. Figure 6 illustrates the choice of collocation points determined to provide adequate resolution for four chordwise vorticity modes ($N = Q = 4$) by five spanwise vorticity modes ($M = 5$) in Equation 4. Adequate resolution was determined by increasing the number of modes and collocation points until the resulting pressure distribution converged to a semblance of the Brown and Michael results(valid for slender wings) for the delta wing (AR=1). Some details of this procedure are presented in Matoi, et al.(1975)[9] for a different set of mode shapes.

An attempt was then made to satisfy the no-force condition in a manner similar to that employed by Nangia and Hancock (1968)[10]. The forces were

calculated using Equation 13, at a set of collocation points, typically five, and the vortex was then moved to reduce these forces, according to the following rule.

$$\frac{d}{dx}\,\Delta y_v = -d\,\frac{F_z}{(F_y^{\,2} + F_z^{\,2})^{1/2}}$$

$$\frac{d}{dx}\,\Delta z_v = d\,\frac{F_y}{(F_y^{\,2} + F_z^{\,2})^{1/2}} \tag{35}$$

where d is chosen small enough to prevent divergence of the procedure, e.g., d = .01. Since the forces have been normalized, the vortex movement is restricted to d. Unfortunately, this method seemed to require considerable discretion in the choice of d. Furthermore, it is necessary to select the number of times to apply the no-force condition, before reapplying the downwash condition. The downwash condition must be satisfied again, since the previous set of vorticity coefficients induces a residual downwash on the wing once the vortex is moved.

One could limit the number of modes in Equation 8 to reduce difficulties with oscillation in the vortex position. However, convergence was not obtained using this procedure, and alternatives had to be considered. The form of Equation 22 suggests that Equation 35 is not the optimum manner for moving the vortex as the velocity components, v_i and w_i, also depend on the vortex location. In the slender-body problem of Brown and Michael, one encounters a similar nonlinear problem for finding the vortex location, y_v and z_v, in the cross-flow plane. Brown and Michael (1955)[4] originally

solved the problem indirectly by assuming values of the vortex location and then satisfying the no-force condition by trial and error. Their problem was greatly simplified in that the downwash condition was automatically satisfied by a conformal transformation, which aligned the two-dimensional flat plate with the flow direction. Later, Pullin (1973)[8] developed a Newton Raphson iteration scheme for the Smith-type model, which included the Brown and Michael problem as a degenerate case. Trial runs of Pullin's program indicated that the Newton-Raphson procedure "converged" in approximately four iterations for the Brown and Michael model.

The Newton-Raphson procedure has several advantages over the procedure developed by Nangia and Hancock which ignores the effect of the change in the vortex position on the velocity components, v_i and w_i. First, the scheme is amenable to automatic iteration without operator interference. Secondly, the iteration procedure converges whenever the derivatives are locally monotonic. Consequently, although the Nangia and Hancock method appeared to work for the simple case they considered, a Newton's method was developed to locate the new vortex position in an iteration procedure. As a preliminary step, a numerical experiment on the applicability of Newton's method was conducted for the slender body model of Brown and Michael, since it was felt that useful information could be obtained on the force Jacobian more economically in two dimensions than in three dimensions. The numerical experiments were conducted on a slender delta wing of unit aspect ratio (AR = 1) at an angle of attack, $\alpha = 14.3°$, which corresponded to the three-dimensional problem being studied. The residual force on the vortex-cut combination was calculated for different vortex locations, which are the unknowns.

In Figure 8 the spanwise force component, F_y is presented, and in Figure 9 the vertical force component, F_z, is given.

The forces are plotted versus the vortex location, y_v and z_v, at $x = 1$. The triangle symbol represents the point where the lines, $F_y = 0$ and $F_z = 0$, intersect to define the stable location for this flow condition. The two-dimensional results indicate that F_z is a monotonic function of both y_v and z_v. F_y, on the other hand, is monotonic in much of the neighborhood of the stable point, but is poorly behaved near the leading edge. This behavior did not preclude the use of Newton's method in the Brown and Michael model, but should be remembered in the event of difficulties in three dimensions.

Therefore, a Newton's procedure was developed for the three-dimensional case to calculate the new vortex location, based on the forces and the force Jacobian calculated in the preceding iteration. This modification improved the rate of convergence in reducing the forces for a given vorticity distribution. However, when the given vorticity distribution was updated to satisfy the downwash condition, the large changes in the vorticity coefficients resulted in large forces. Various attempts to limit the changes in the vorticity coefficients and the vortex location coefficients were made by only partially reducing the forces and then partially reducing the residual downwash in a sequential procedure. This procedure did not appear to be converging; so a more detailed look was taken of the slender-body problem.

Since the downwash condition can not be automatically satisfied in the three-dimensional case as in the slender-body case, this difference may hide the cause of the convergence difficulties. Therefore, the two-dimensional problem was investigated in a manner parallel to the three-dimensional problem.

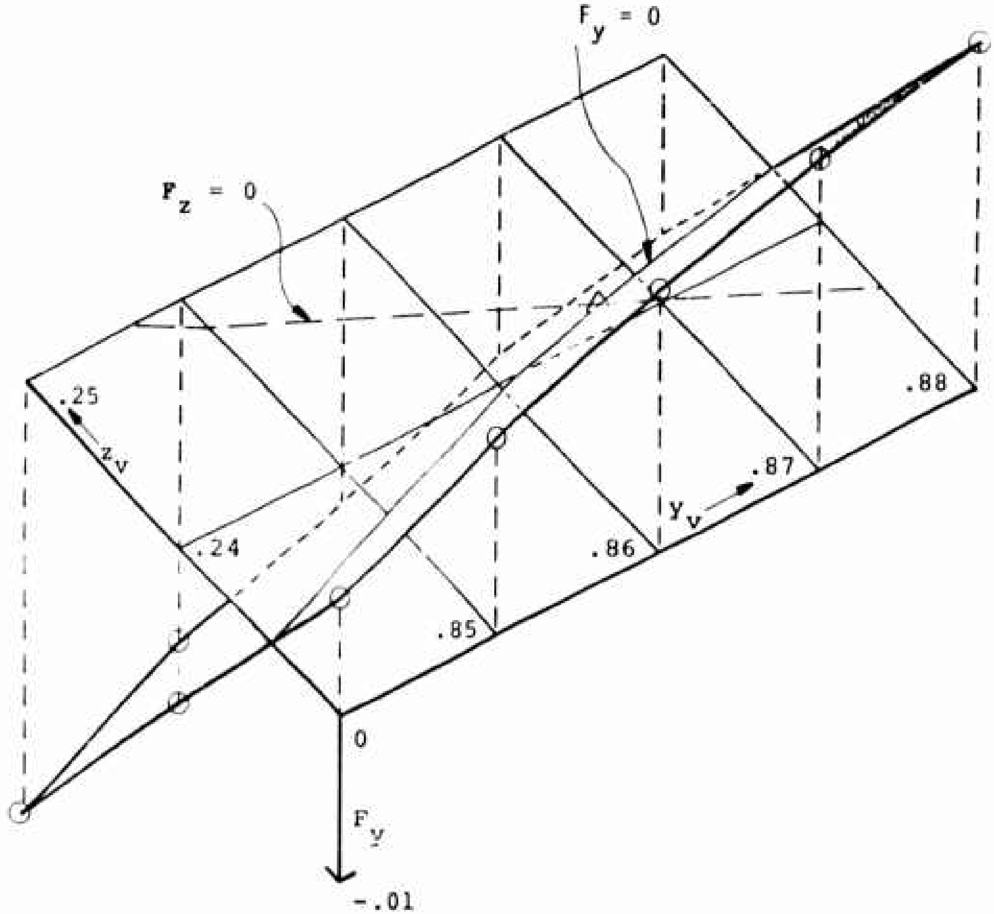


Figure 8. Spanwise force component on leading-edge vortex versus vortex location. Downwash condition satisfied on delta wing ($\sin\alpha/\cot\lambda = 1$) for Brown and Michael model. Symbol (Δ) represents stable point, $F_z = F_y = 0$.

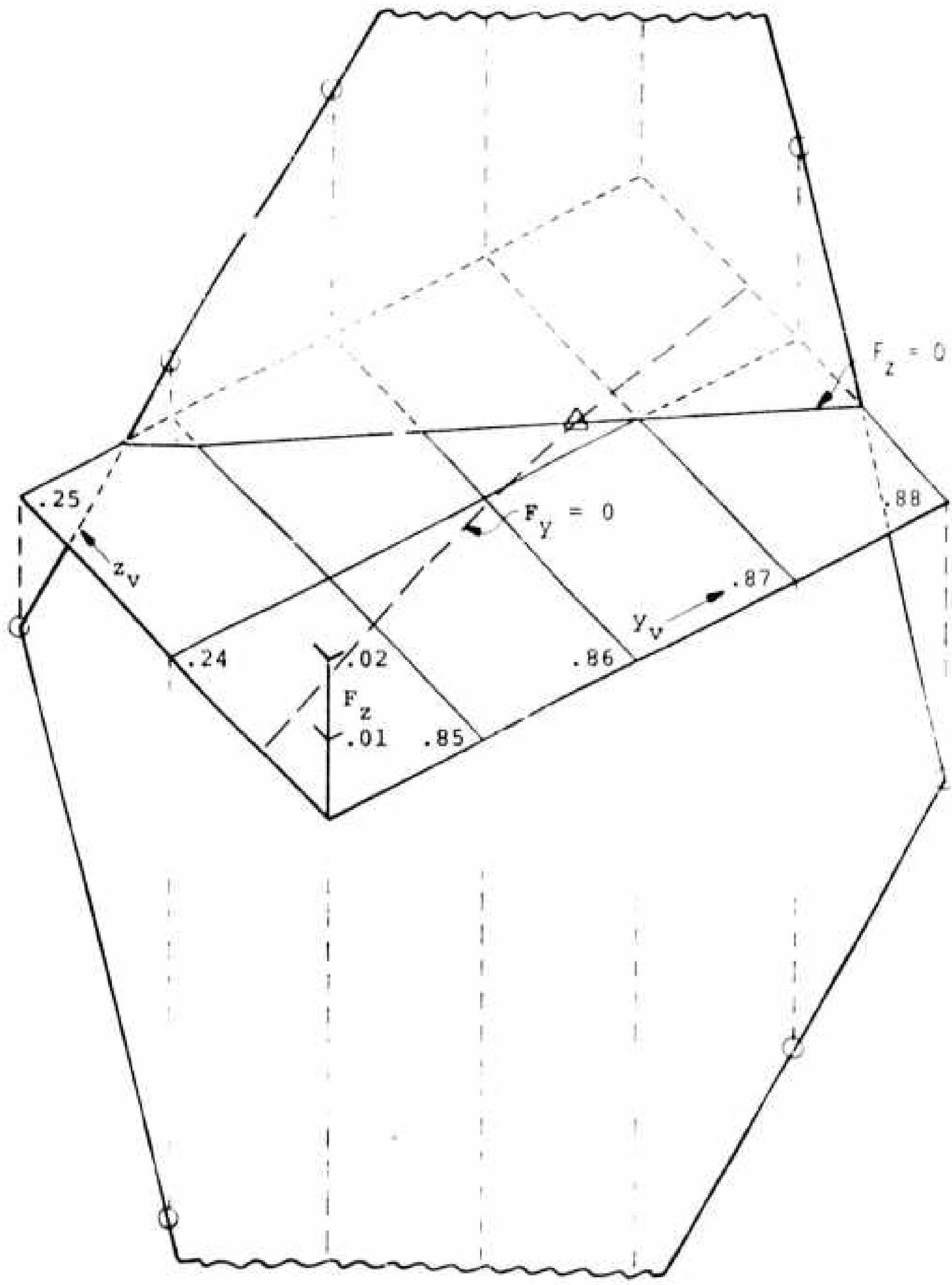


Figure 9. Vertical force component on leading-edge vortex versus vortex location. Downwash condition satisfied on delta wing ($\sin\alpha/\cot\lambda = 1$) for Brown and Michael model. Symbol (Δ) represents stable point, $F_y = F_z = 0$.

The Brown and Michael problem was thus reformulated as a vorticity distribution on the wing with unknown loading coefficients. The leading-edge vortex strength and location were also unknown originally. The downwash and no-force condition were then written in terms of these unknowns in the physical y-z plane.

The known location of the vortex was first used to calculate the vorticity coefficients from the downwash condition. Then the vortex was moved to different points, and the resulting forces are plotted in Figure 10 and Figure 11. Although the vertical force component appears similar to the one obtained previously (cf., Figure 9), the spanwise force component has changed considerably (cf., Figure 8). Especially significant is the fact that the lines, $F_y = 0$ and $F_z = 0$, are nearly coincident, which suggests difficulties in finding their point of intersection. In fact, when an attempt was made to iterate between reducing the downwash residue and the forces on the vortex, the procedure failed to converge. The procedure oscillated between the true solution and a false solution, where the forces were zero, but the downwash condition was not satisfied. Some of the details of these calculations are included in Appendix B.

Therefore, an alternative strategy was developed, whereby the forces and downwash residues were reduced simultaneously, instead of sequentially, by changing the vorticity coefficients as well as the vortex location according to Newton's method. This procedure, although requiring more effort to calculate the derivatives of the downwash terms as well as the derivatives of the force terms with respect to the vorticity coefficients, resulted in smooth convergence to the proper solution. As a typical example, five vorticity modes and six control points were employed for the slender delta

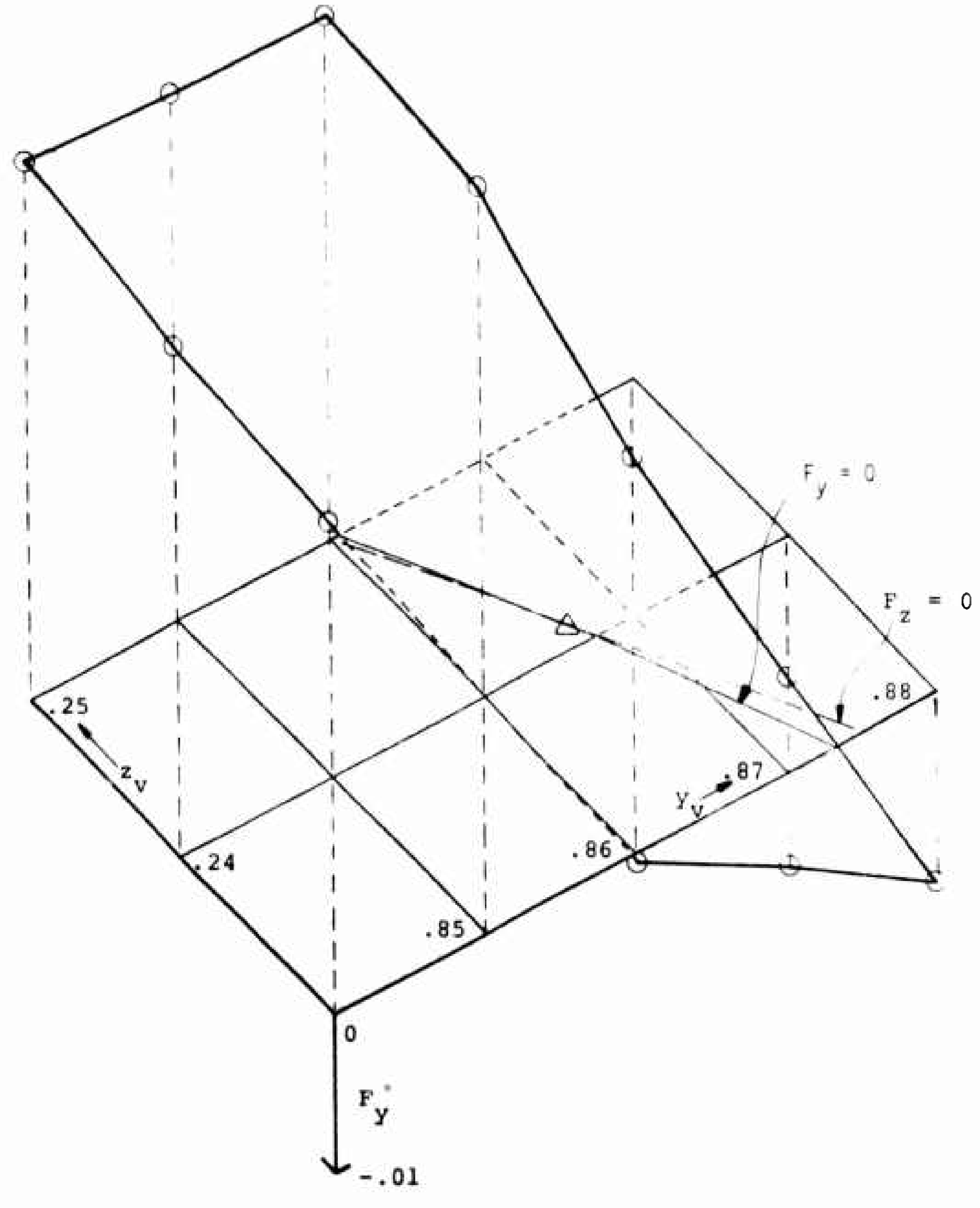


Figure 10. Spanwise force component on leading-edge vortex versus vortex location for modified Brown and Michael model. Vorticity coefficients chosen to satisfy downwash condition at stable point for delta wing ($\sin\alpha/\cot\lambda = 1$). Symbol ($\Delta$) represents stable point, $F_z = F_y = 0$.

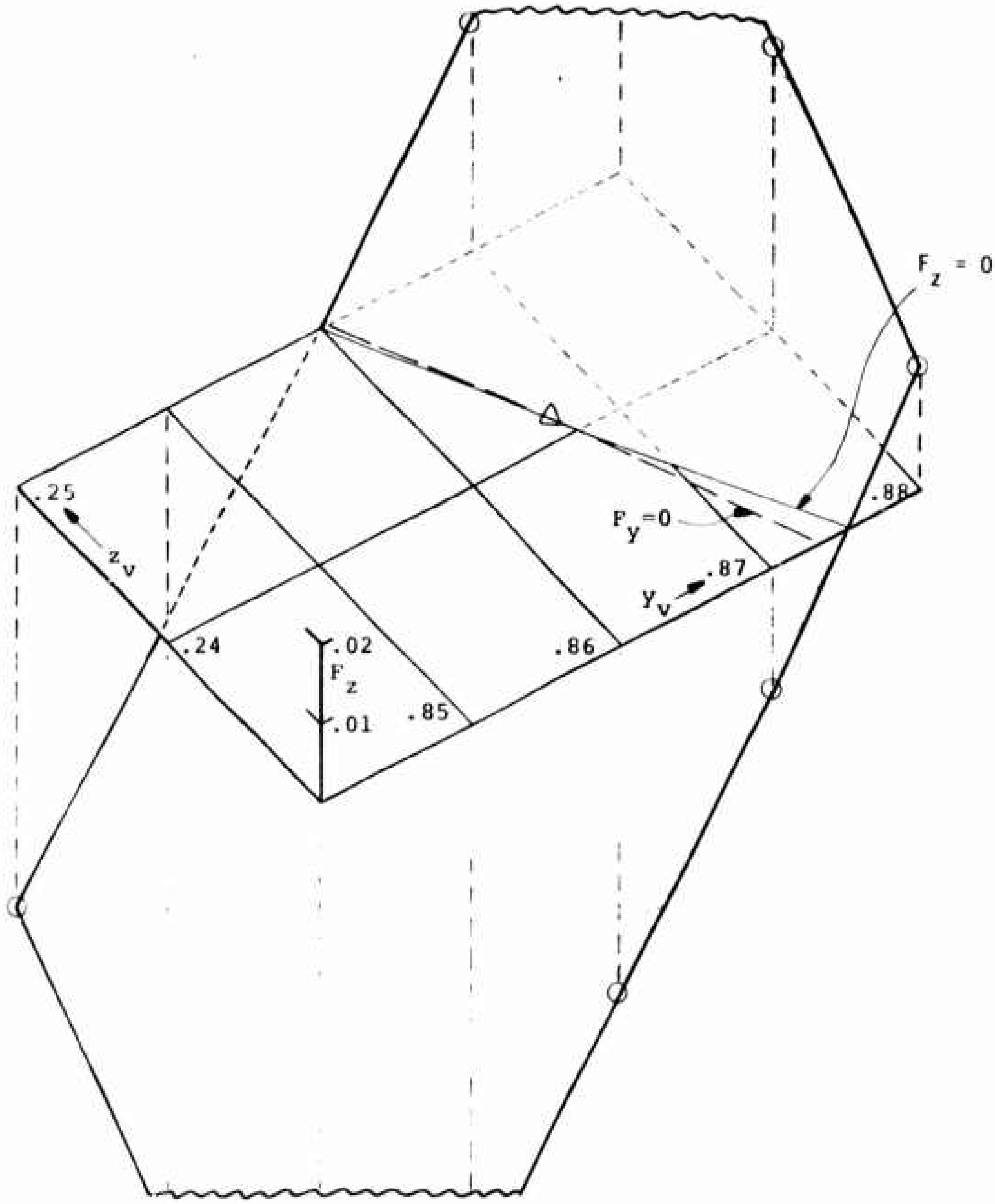


Figure 11. Vertical force component on leading-edge vortex versus vortex location for modified Brown and Michael model. Vorticity coefficients chosen to satisfy downwash condition at stable point for delta wing ($\sin\alpha/\cot\lambda = 1$). Symbol ($\Delta$) represents stable point, $F_y = F_z = 0$.

wing (AR = 1, α = 14.3°) being considered. The vortex was initially assumed to have a spanwise location of 80 per cent of the semispan and a height of 30 per cent of the semispan (y_v/s = .8, z_v/s = .3). The iteration procedure converged to a stable point (y_v/s = .86, z_v = .24) in eight iterations. See Figure 12 for a graphic description of the convergence rate. Thus, the procedure was adapted to the fully three-dimensional case.

The full procedure presently being employed to satisfy the downwash and no-force condition is described next. First, an initial location for the vortex is found. This initial location is used in conjunction with Equation 20 to find an initial distribution of vorticity coefficients. Now, the residual forces and the remaining derivatives for the Jacobian are calculated. The residues and the Jacobian are used to calculate a new set of vorticity coefficients and vortex location coefficients. This last step is iterated until the procedure converges. If the same number of equations and unknowns are employed, then convergence is attained when the residue becomes small compared to the angle of attack. If the number of equations is greater than the number of unknowns, it is generally impossible to satisfy all of the conditions imposed, and convergence is attained when the residue is minimized and further iterations produce no additional change. Although the Jacobian is presently being updated for the force contributions at every iteration, it appears that some savings in computational effort may be obtained by only a partial updating as most of the derivatives change slowly. This modification can be implemented after greater knowledge of the procedure has been acquired.

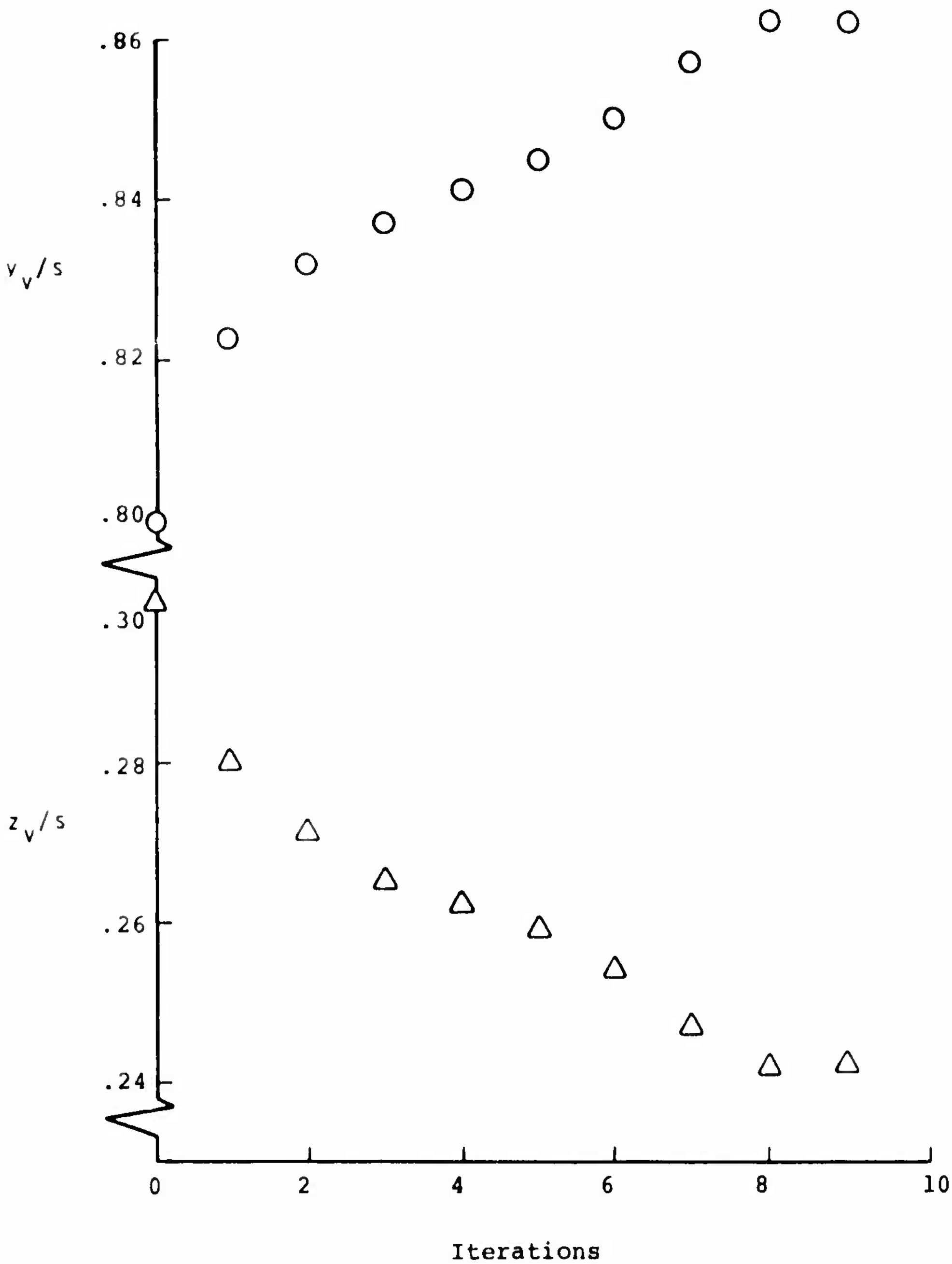


Figure 12. Convergence of vortex location to stable point in cross-flow plane for slender delta wing (AR = 1, α = 14.3°).

4. Program Description

The actual FORTRAN programs to perform the operations described in the previous section are included in Appendix C and are documented primarily by comment cards. Additionally, the coded symbols are generally similar to their English counterparts to facilitate comprehension. The complete procedure is presently divided into five computer programs: Program I, Program WOW, Program IIIA, Program III Prime, and Program V.

Program I calculates the influence coefficients due to the contributions from γ_2 and δ_2 for the downwash condition at a set of collocation points for the specified number of chordwise modes. The number of chordwise modes, Q and N, in Equation 4 is represented by the FORTRAN variable NOCM. The number of spanwise modes, M, in Equation 4 is represented by the variable NOSM. The number of chordwise collocation stations on the wing surface is given by NCORD and the number of spanwise collocation stations is given by NSPAN, where the product of these numbers (NCORD times NSPAN) must be greater than or equal to the total number of modes (NOCM times (NOSM + 1) for the system to be completely determined.

Program WOW is the program which evaluates the contribution of γ_1 to the downwash condition according to Equation 10, at the chosen set of collocation points. As mentioned previously, this is a simplified version of the program developed by Widnall (1964)[13] to calculate the influence coefficients from a distribution of horseshoe vortices.

Program IIIA uses the results of Program I and Program WOW as inputs and, furthermore, calculates the contributions from the leading-edge vortices and wake due to δ_2. Then it solves a set of simultaneous equations based on the downwash condition (Equation 20), to find the initial values for the

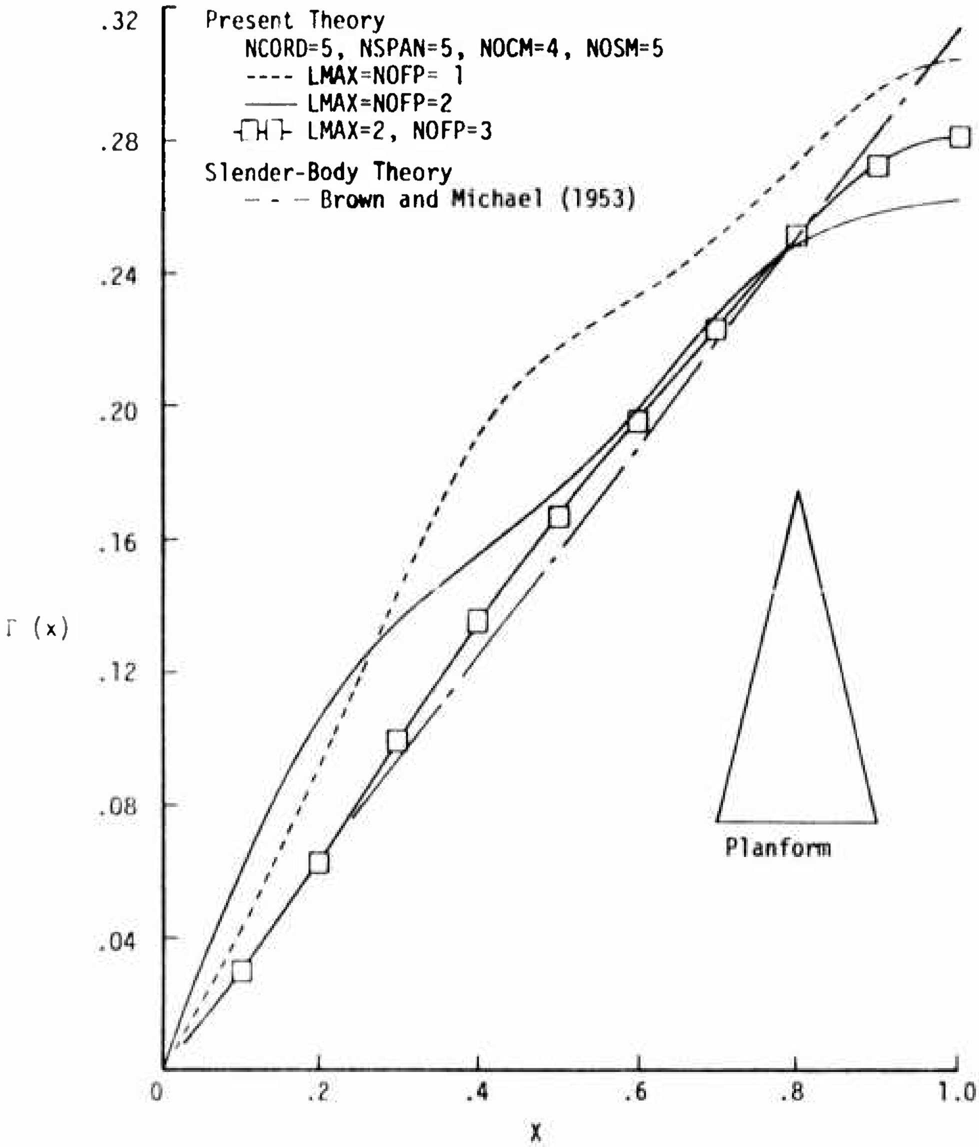


Figure 13. Convergence of leading-edge vortex strength for delta wing ($AR = 1$, $\alpha = 14.3^{\circ}$).

near the apex. This is in agreement with experiments which generally show that the flow approximately satisfies the Brown and Michael conditions, away from the trailing edge. Three-dimensional effects are apparent in the slope of the leading-edge vortex strength. The modes have been chosen to insure that the slope is zero at the trailing edge, after which no additional vorticity is shed from the leading edge.

Figure 14 illustrates the change in the stable position of the leading-edge vortex on the right half of the wing. The spanwise position from the Brown and Michael model is not included since it is almost coincident with the NOFP = 1, LMAX = 1 result. The parameter choice, LMAX = 1 (see Equation 8 for details of the expansion), corresponds to a linear approximation for the vortex position, while the choice, LMAX = 2, represents a cubic fit for the vortex location. As can be seen from Figure 14, the spanwise position changes slightly, although the three-dimensional effect seems to be manifested in an effort to align the vortex with the free stream direction. The same tendency is indicated by the vertical position of the vortex, although convergence is only partly indicated by the bracketing of the final vortex location by the lower order models.

A comparison with the experimental results of Peckham (1958)[15] and with some slender-body models is presented in Figure 15 for the vortex position over the delta wing. The agreement for the vertical position is excellent, while the spanwise position indicates the general limitations of a Brown and Michael model in predicting the vortex location too far outboard. It must be noted that this is not a completely fair test, since the vortex location represents the center of vorticity in the Brown and Michael model. For the Smith-type model,

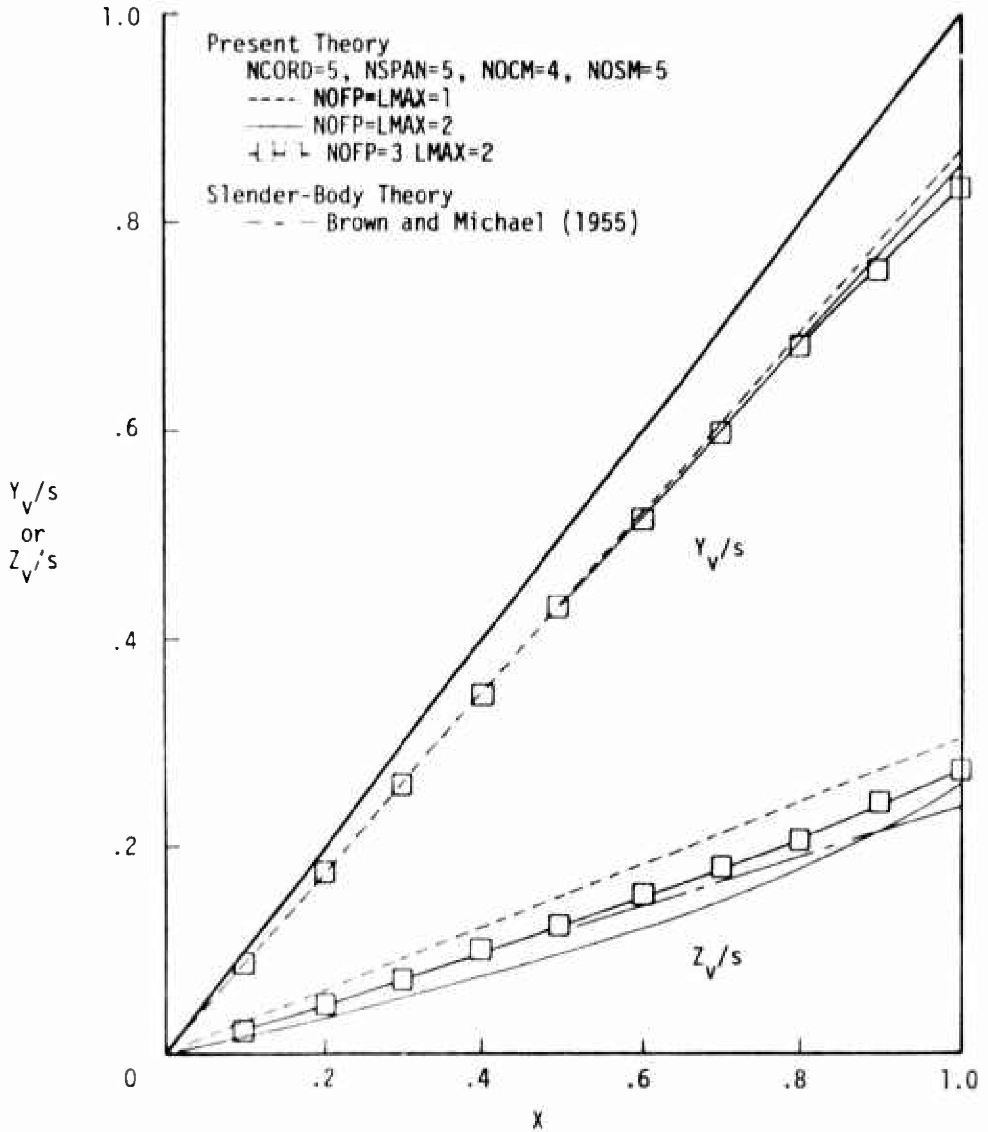


Figure 14. Convergence of vortex position over delta wing (AR=1, α =14.3^0).

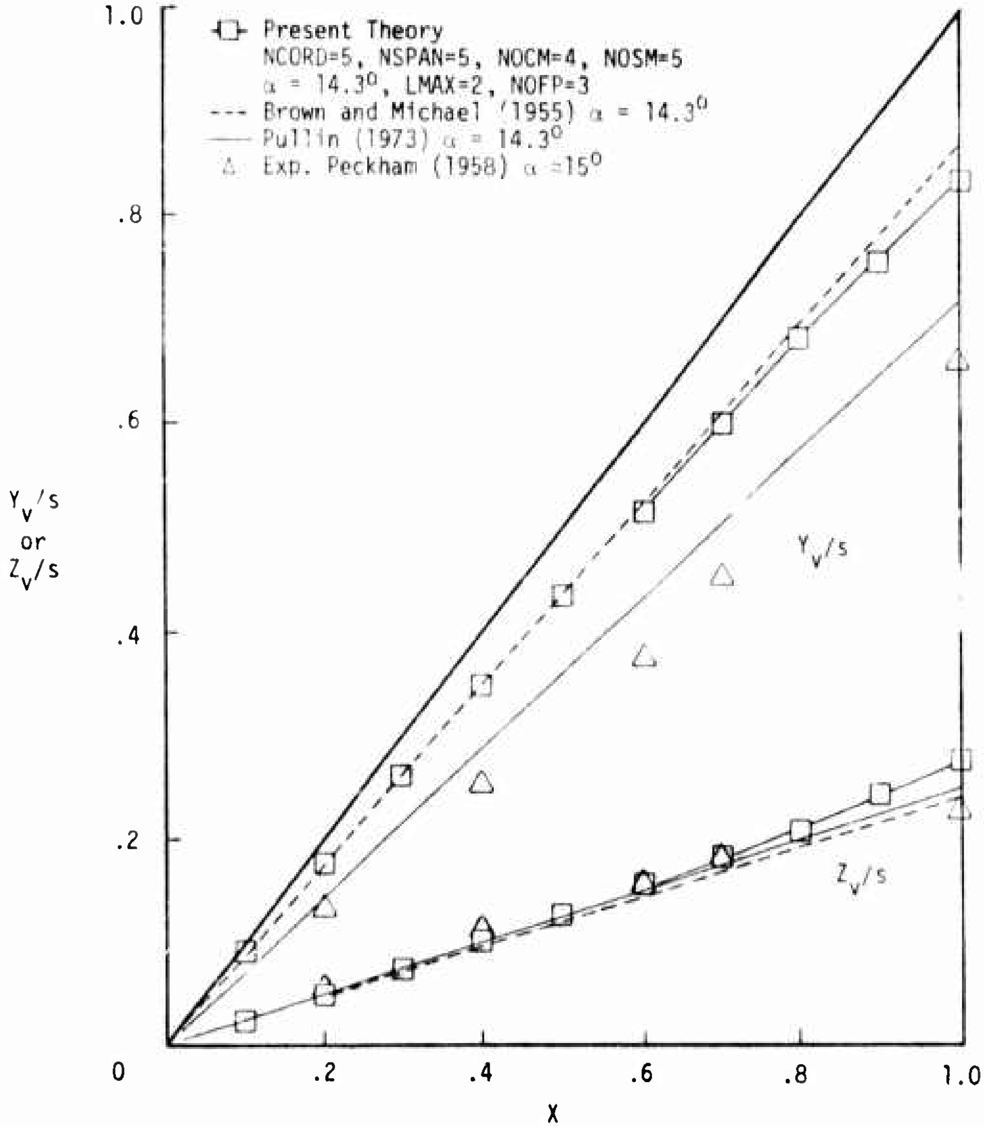


Figure 15. Leading-edge vortex position over right half of delta wing (AR=1).

on the other hand, the spanwise location of the center of vorticity is five per cent of the semispan outboard of the core position. This shift is due to the presence of vorticity in the leading-edge vortex sheet. Thus, it would be reasonable to assume that the spanwise location of the center of vorticity for the experimental data is also outboard of its vortex core location.

Some pressure distributions are presented next. In Figure 16, the pressure distribution calculated by the present procedure is compared with the results of the slender-body models. Again excellent agreement is obtained with the Brown and Michael model for the stations near the apex while the aft stations show the attenuation due to the presence of the trailing edge.

Figure 17 shows a comparison with the experiments of Nangia and Hancock (1969)[16] on a flat plate delta wing. The experimental results are presented as a smooth curve as provided in the referenced report. The general shape and magnitude of the loading have been predicted, but the limitations of a Brown and Michael model are again apparent. The predicted peak is too far outboard and too high.

Figure 18 presents a comparison with some data available for a thick delta wing. Here the thickness to chord ratio is .12, and a comparison of lift curves from Peckham (1958)[15] indicates that the pressure peaks are ten to twenty per cent lower for the thick wing than for the flat plate wing. Also the vortex position is further inboard and higher for the thick wing. These effects have been verified theoretically in the slender-body range by Smith (1971)[17]. Thus, detailed comparison between the experimental data of Peckham for thick wings and the theoretical predictions for flat plate delta wings is limited.

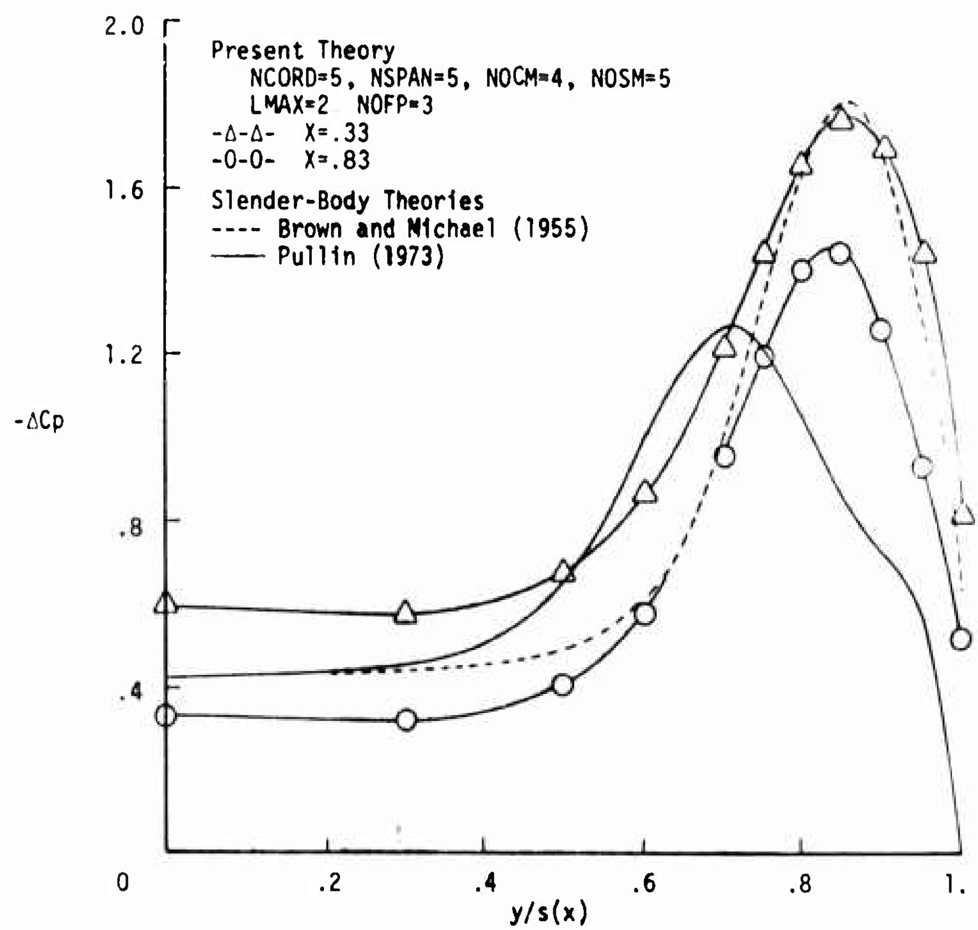


Figure 16. Comparison of theoretical models for calculation of loading on delta wing (AR=1, α = 14.3°).

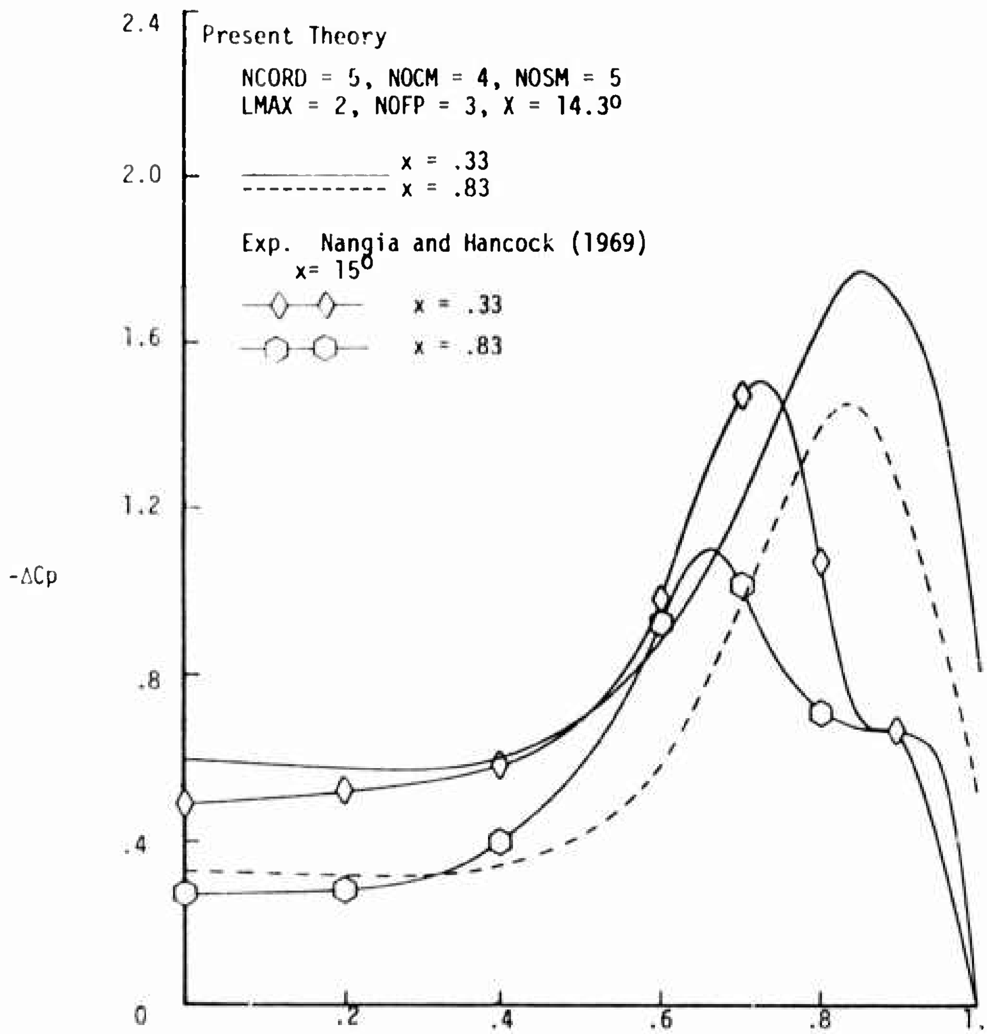


Figure 17. Comparison of experimental and theoretical loading values on delta wing (AR=1).

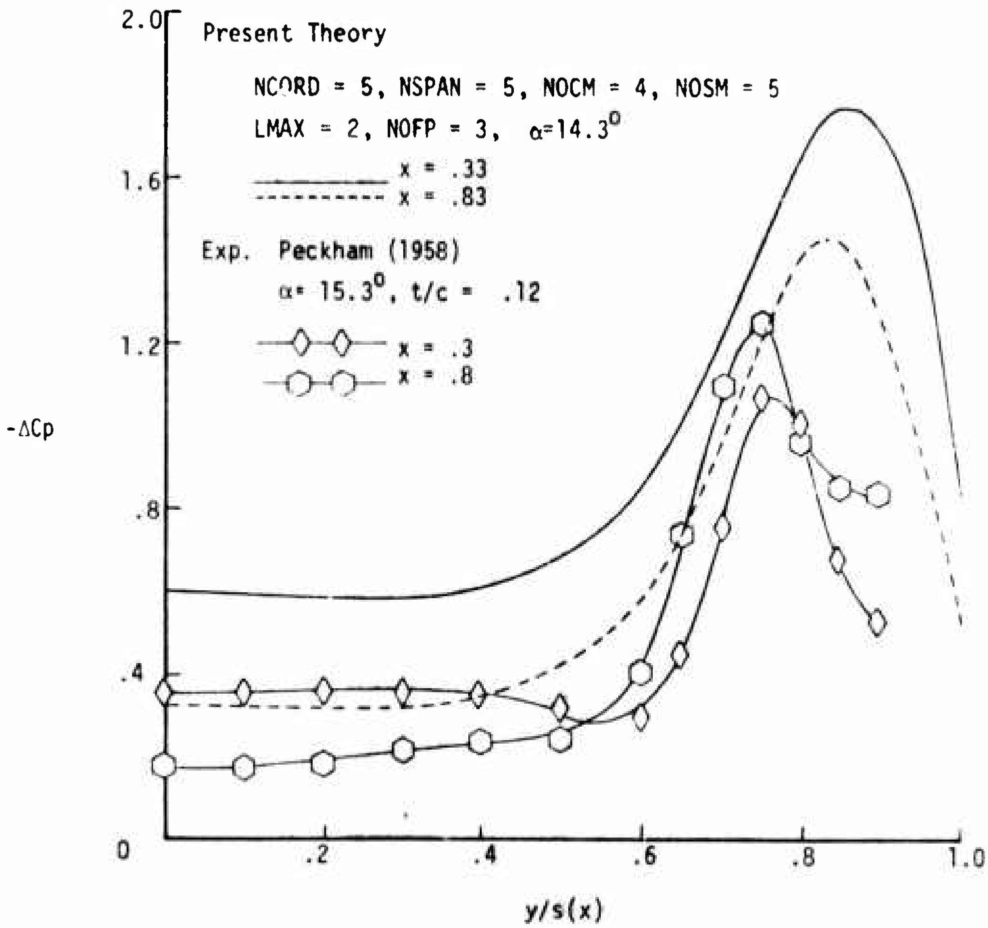


Figure 18. Comparison of experimental and theoretical loading values on delta wing (AR=1).

Finally, a comparison is made with some other lifting surface theories. As mentioned in the Introduction, Brune, et al.(1975)[8] and Kandil, et al. (1974)[7] have developed finite-element lifting surface theories with leading-edge separation. A comparison of these theories with the present theory and the experimental results of Peckham(1958)[15] is presented in Figure 19. Both of the other theoretical curves were taken from Kandil, Mook, and Nayfeh (1976)[18], who referenced Weber, Brune, Johnson, Lu, and Rubbert (1975).[19] As expected, the present theory, which is based on a Brown and Michael vortex-cut representation predicts a higher peak loading which is further outboard than the one predicted by the other lifting surface theories for the flat plate delta wing. However, since the experimental curve is for a 12 per cent thick wing, one would expect the experimental pressure peak to be higher and further outboard if the wing were thin, for the reasons presented during the discussion of Figure 18. Thus, although the present theory does not provide solutions identical with those provided by the other theories, the present procedure appears to be competitive in predicting the experimental results compared with the other programs.

In Figure 20, the results for the sectional normal force coefficients are compared with those obtained by Nangia and Hancock (1969).[16] The sectional normal force coefficient is defined as

$$C_N\,(x) = \int_{-s(x)}^{s(x)} \Delta\, Cp\; dy \qquad (36)$$

Again, the tendency of the Brown and Michael model to overpredict the magnitude of the loading is apparent. Although the sectional force coefficient calculated by the present method decreases near the trailing

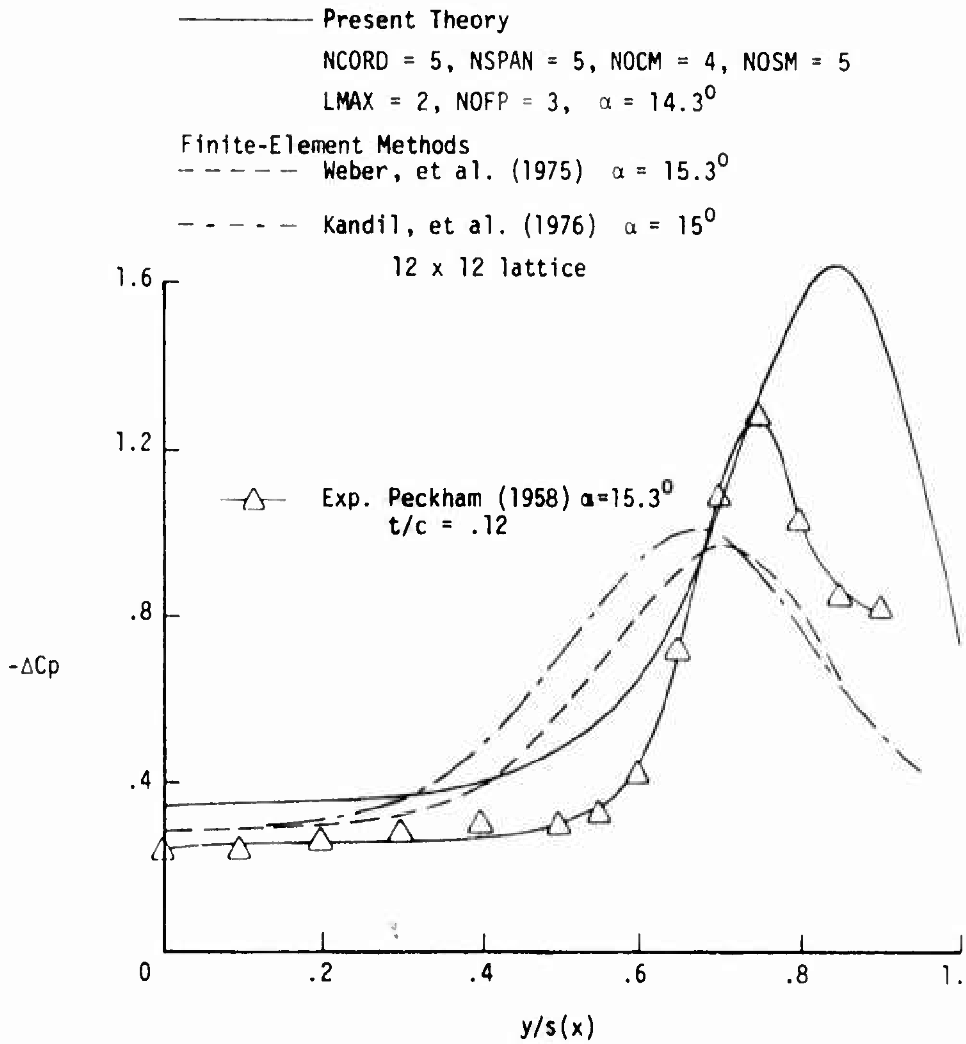


Figure 19. Comparison of lifting surface models for the calculation of loading on delta wing (AR=1, X = .7).

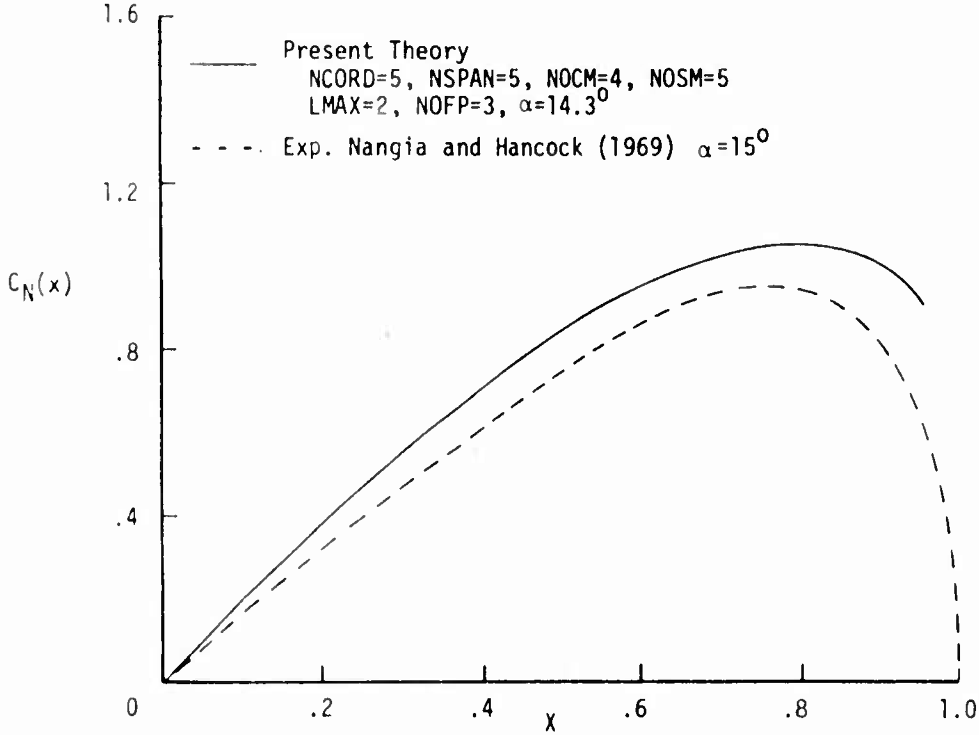


Figure 20. Chordwise distribution of sectional normal force coefficients for delta wing (AR=1).

edge, it does not vanish since a modified Kutta condition has been applied which does not require zero loading at the trailing edge.

To demonstrate that this program could be used for planforms other than the delta wing, some runs were made for the arrow wing. However, the problem with this planform and others is that there is little experimental evidence readily available for these planforms.

Figure 21 illustrates the results for the leading-edge vortex strength for an arrow wing whose planform is similar to that of the unit aspect ratio delta wing with the addition of trailing edge sweep ($\lambda = 76^{\circ}$, AR = 1.25) at an angle of attack $\alpha = 14.3^{\circ}$. The same number of collocation points for the downwash and the same number of vorticity modes were used as for the delta wing (NCORD= 5, NSPAN = 5, NOCM = 4, NOSM = 5). The initial approximation for the vortex location was obtained from the delta wing being considered previously. It appears that convergence is more difficult to obtain for the arrow wing than for the delta wing as an extraneous "bump" appears in the curve for the case NOFP = 2, LMAX = 1. This is smoothed over as an additional constraint is applied. Near the apex, the vortex strength is similar to that for a delta wing of unit aspect ratio, as would be expected away from the trailing edge.

The vortex position for this arrow wing is plotted in Figure 22. The results for the cubic fit (LMAX = 2) with three no-force points (NOFP = 3), are similar to those obtained for the delta wing, indicating the dominance of the leading-edge sweep in locating the vortex.

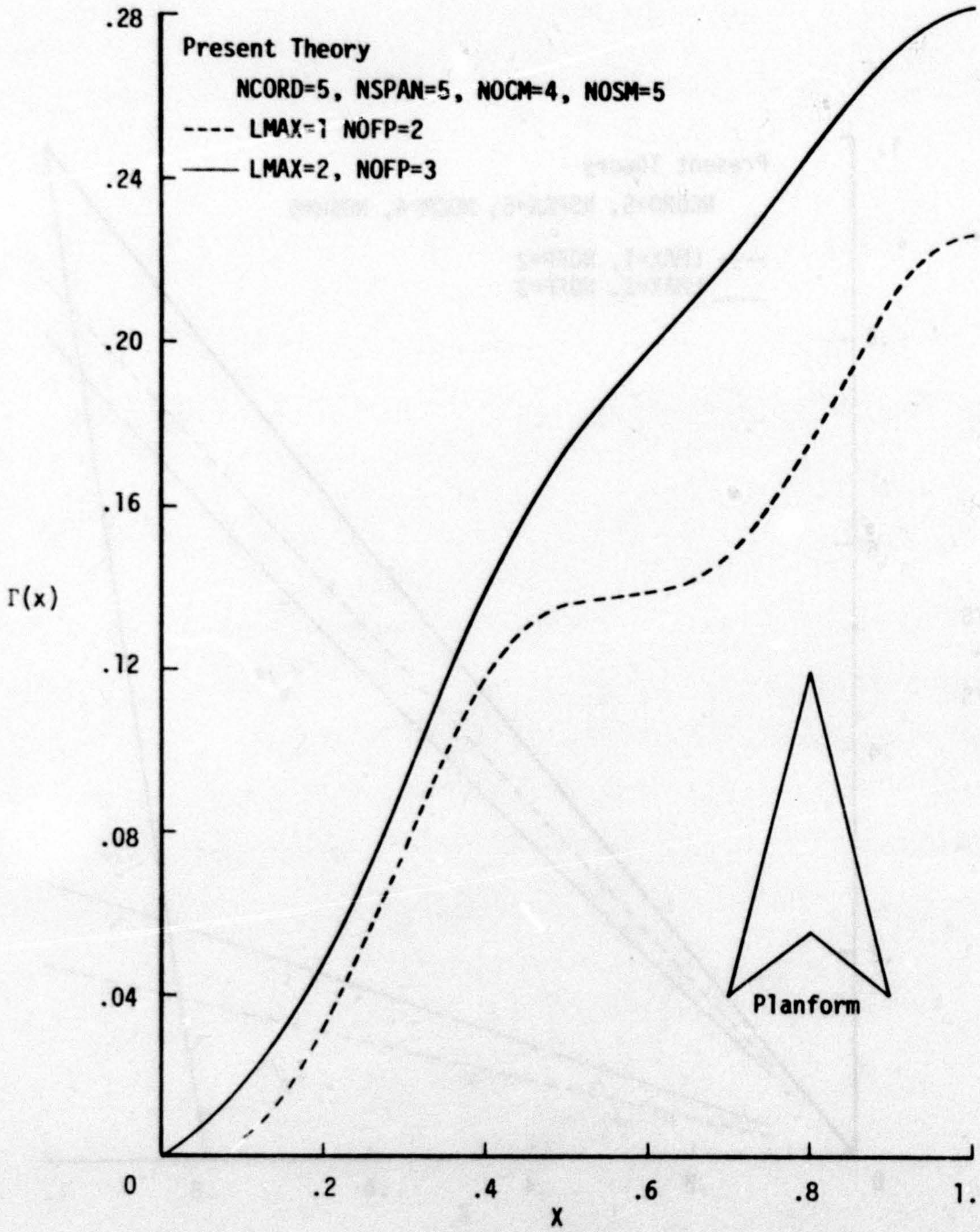


Figure 21. Convergence of leading-edge vortex strength for arrow wing (AR=1.25, $\lambda=76^0$, $\alpha=14.3^0$).

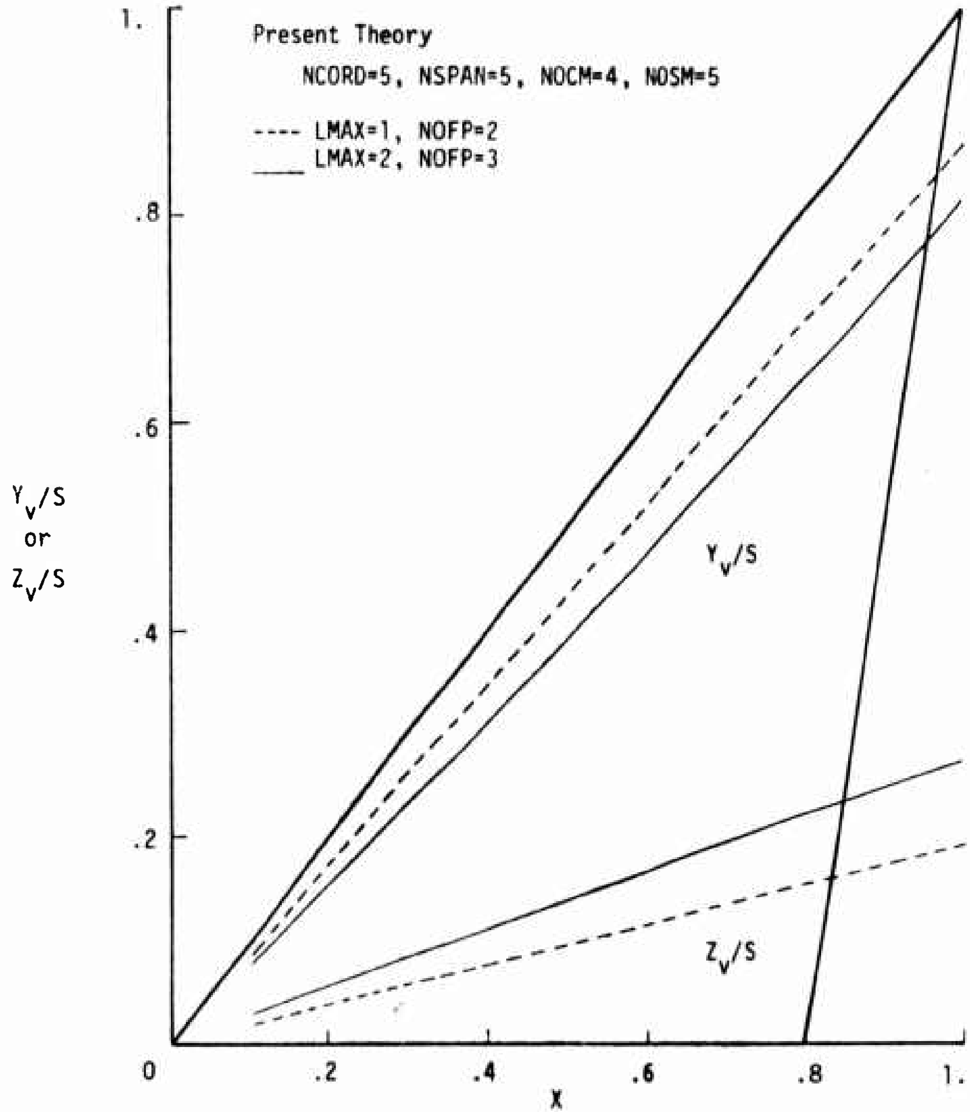


Figure 22. Leading-edge vortex position over right half of arrow wing (AR=1.25, $\lambda = 76^0$, $\alpha = 14.3^0$).

Finally, the pressure distributions at two chordwise stations are plotted in Figure 23. It is to be noted that the station, $x = .833$, is aft of the root chord ($x = .8$) and consequently, the loading should strictly be zero at both the trailing edge ($y/s(x) = .2$) and at the leading edge ($y/s(x) = 1$).

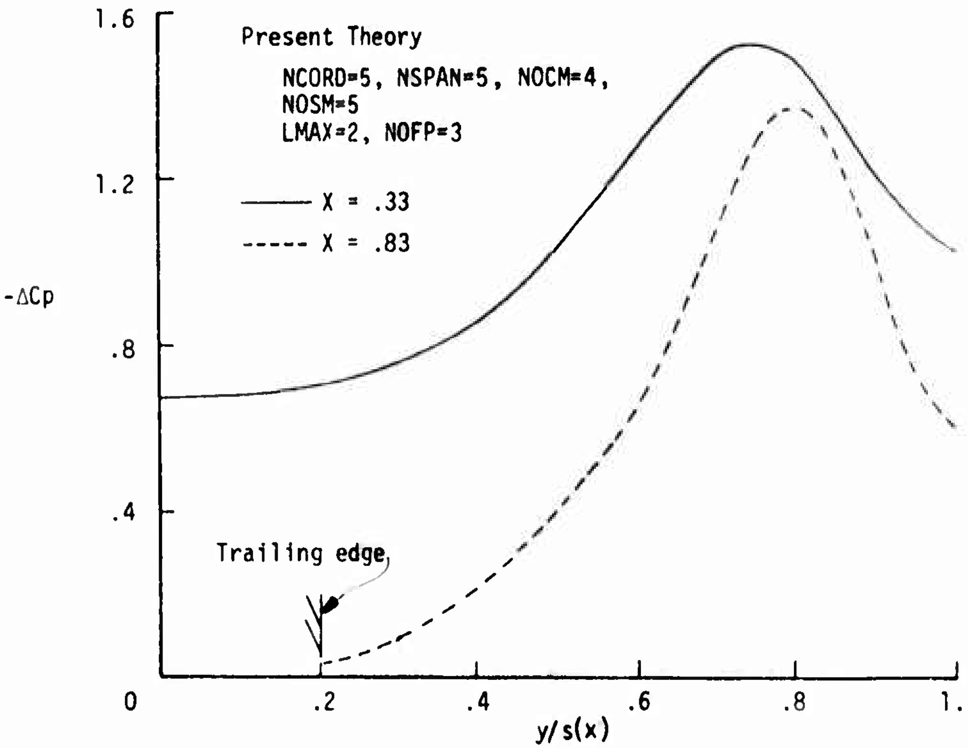


Figure 23. Loading on arrow wing (AR= 1.25, λ =76^0, α = 14.3^0).

6. Revised Vorticity Modes

The results obtained for the arrow wing in the previous section demonstrated the need for a better representation of the vorticity distribution at the trailing edge for such reentrant surfaces. Problems with convergence in the iteration procedure were encountered as the trailing edge sweep angle was increased. The reason for this difficulty can probably be traced to the form of bound vorticity chosen.

The present model used bound vorticity modes to feed the leading-edge vortices which are circular arcs with their center at the apex of the wing (see Figure 5). However, at the trailing edge, this vorticity was suddenly turned downstream as described in Figure 4. Consequently, there was a discontinuous turning of the vortex lines at the trailing edge. This was not a serious problem for the slender delta wing, where the choice of vorticity modes insured that the spanwise component, γ, was small compared to the chordwise component, δ, at the trailing edge. However, as the sweep angle of the trailing edge was increased, a sharp kink developed in the bound vorticity at the trailing edge due to the nonzero component, γ_2, which fed the leading-edge vortices. No control points were located at the trailing edge, so no singularities were encountered in the numerical calculations, but such a discontinuity was a potential source of trouble.

An effort was made to develop a better modal description of the bound vorticity which feeds the leading-edge vortices, i.e., γ_2 and δ_2 The desired conditions to be satisfied by these vorticity components on the right half of the wing are

$$\gamma_2(X_{TE}(y),y) = 0 \tag{37}$$

$$\frac{\delta_2(X_{LE}(y),\ y)}{\gamma_2(X_{LE}(y),y)} = -s \tag{38}$$

where Equation 37 guarantees that there is no kink at the trailing edge and Equation 38 insures that the vorticity leaves perpendicular to a straight leading edge. It is to be noted that symmetry conditions dictate that γ_2 is even and δ_2 is odd in the spanwise variable.

An attempt was first made to determine vorticity functions which satisfied these conditions in the physical (x,y) plane. However, that approach failed to provide a solution, and the problem was then considered in the transformed (θ,η) plane. (See Equation 5 for the coordinate transformation.) Basically, in the transformed plane, the leading edge of the planform corresponds to the chordwise origin, $\theta = 0$, and the trailing edge corresponds to the chordwise maximum, $\theta = \pi$.

The two vorticity components, γ_2 and δ_2, can be written in the following form.

$$\gamma_2 = \sum_{q=1}^{NOCM} g_q\, g_\gamma\ (\eta,q)$$

$$\delta_2 = \sum_{q=1}^{NOCM} g_q g_\delta\ \ (\eta,q) \tag{39}$$

Then, from the continuity of vorticity, Equation 2, the two functions can be related by

$$g_\delta = -\frac{1}{2s}\int^{\theta} \frac{\partial g_\gamma}{\partial \eta}\ c(\eta)\ \sin\theta d\theta \tag{40}$$

Assuming a form which satisfies the boundary condition at the trailing edge and is nonzero at the leading edge, let

$$g_\gamma = \cos\theta/2 \;\; f(\eta,q) \qquad (41)$$

For the arrow wing planform being considered presently, the local chord on the right-hand side is

$$c(\eta) = c_R\,(1-\eta)$$

where c_R is the root chord nondimensionalized by the maximum length. Then Equation 40 becomes

$$g_\delta = \frac{c_R}{6s}\left\{ \frac{\partial f}{\partial \eta}(1-\eta)\,[3\cos\theta/2 + \cos 3\theta/2] \right.$$

$$\left. + \frac{1}{2} f(\eta,q)\left[\frac{3(4-3c_R)}{c_R}\cos\theta/2 + \cos 3\theta/2\right] \right\} - g(\eta,q) \qquad (42)$$

where the function, $g(\eta,q)$, is a function of integration.

Using the identity

$$\cos\frac{3\theta}{2} = 4\cos^3\theta/2 - 3\cos\theta/2$$

this can be rewritten as

$$g_\delta = \frac{c_R}{3s}\left\{ 2\frac{\partial f}{\partial \eta}(1-\eta)\cos^3\theta/2 + f(\eta,q)\left[\frac{3(1-c_R)}{c_R}\cos\theta/2 + \cos^3\theta/2\right]\right\} - g(\eta,q) \quad (43)$$

To determine the function, $f(\eta,q)$, it is necessary to apply the boundary condition at the leading edge, Equation 38.

$$-s = \frac{\frac{c_R}{3s}\left\{2\frac{\partial f}{\partial \eta}(1-\eta) + \frac{3-2c_R}{c_R} f(\eta,q)\right\} - g(\eta,q)}{f(\eta,q)} \tag{44}$$

This provides the following differential equation

$$\frac{\partial f}{\partial \eta} + \frac{1}{2(1-\eta)c_R}\left[3(s^2+1) - 2c_R\right] f(\eta,q) = \frac{3s\, g(\eta,q)}{2(1-\eta)c_R} \tag{45}$$

The solution of this differential equation is

$$f(\eta,q) = C(1-\eta)^{\frac{3(s^2+1)}{2c_R} - 1} +$$

$$\frac{3s(1-\eta)}{2\, c_R}^{\left[\frac{3(s^2+1)}{2c_R} - 1\right]} \int_1^{\eta} (1-\eta)^{-\frac{3(s^2+1)}{2\, c_R}} g(\eta,q)\, d\eta \tag{46}$$

where C is a constant of integration. In order to obtain a general modal description, one must allow $g(\eta,q)$ to be a complete set of functions. The constant, C, is chosen to be zero, while the following form is chosen for $g(\eta,q)$ to simplify the integration

$$g(\eta,q) = \frac{(1-\eta)^q}{3s} \tag{47}$$

Then, integration of Equation 46 yields

$$f(\eta,q) = \frac{-(1-\eta)^q}{2c_R(q+1) - 3(s^2+1)} \tag{48}$$

Therefore, the vorticity functions become

$$g_\gamma = \frac{-(1-\eta)^q}{2c_R(q+1) - 3(s^2+1)} \cos \theta/2$$

$$g_\delta = \frac{1}{3s}(1-\eta)^q \left\{ \frac{(2q-1)\, c_R \cos^3\theta/2 + 3(c_R-1)\cos\theta/2}{2c_R(q+1) - 3(s^2+1)} - 1 \right\} \quad (49)$$

These new representations are used to replace the γ_2, δ_2 contributions in the previous calculations. They have the advantage over the previous functions in that there is no longer a kink in the vortex lines, as γ_2 now vanishes smoothly at the trailing edge.

The programs previously described in Section 4 have been modified to include these new vorticity modes, but time limitations have restricted the investigation with these new modes. After several preliminary runs were made for the unit aspect ratio delta wing to determine convergence and resolution factors, the following parameters were adopted as adequate to describe the bound vorticity modes. Five spanwise and five chordwise stations (NSPAN = 5, NCORD = 5) have been employed, and three chordwise modes (NOCM = 3) and four spanwise modes (NOSM = 4) have been selected.

The planform considered was that of the arrow wing employed by Brune, et al. (1975)[8] to test their lifting surface theory program, based on finite element panels. The wing has a leading-edge sweep angel, $\lambda = 71.2^0$, an aspect ratio, AR = 2.02, and an angle of attack, $\alpha = 15.8^0$. Convergence for the leading-edge vortex strength for various numbers of no-force points (NOFP) and degrees of freedom in the vortex location (LMAX) are presented in Figure 24. In comparison with

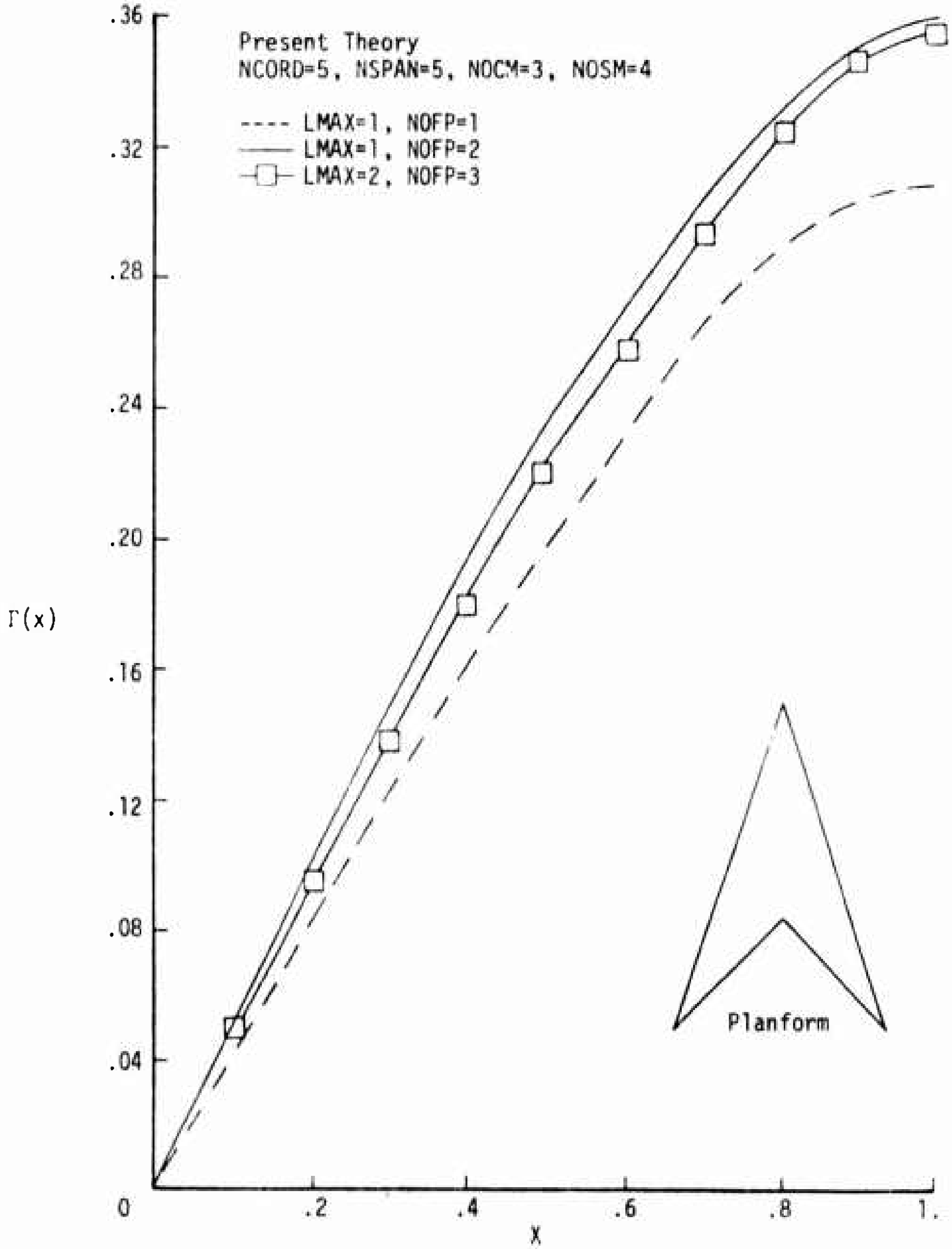


Figure 24. Convergence of leading-edge vortex strength for arrow wing (AR=2.02, λ = 71.2°, α = 15.8°)

Figure 21, which provided results for an arrow wing using the earlier vorticity modes, it is apparent that the new vorticity modes result in much smoother convergence. This is true even though a larger traiing-edge sweep angle is being considered now than before. Again the modes have been chosen to provide no additional feeding of vorticity from the leading edge, aft of the trailing edge. Thus, the slope of the circulation strength of the leading-edge vortex vanishes at the trailing edge.

The variation of the vortex position over the right half of the wing is presented in Figure 25 as a function of the number of force points and degrees of freedom in the vortex location. The original forms for the vortex position modes (see Equation 8) have been used and the choice, LMAX = 1, corresponds to a linear fit, while the selection, LMAX=2, corresponds to a cubic approximation for the vortex position. The vortex position obtained by this numerical procedure appears quite stable even with these few no-force points. For the cubic approximation, the apparent tendency of the leading-edge vortex to align itself with the free stream direction near the trailing edge is noted.

Finally, the pressure distributions predicted by the present program are compared in Figure 26 with the results of Brune, et al. (1975)[8] at two chordwise stations. Brune, et al. employed 30 wing panels, and 48 free-vortex-sheet panels - each vortex-sheet panel contributed two unknowns since both its strength and orientation were originally unknown - for a total of 126 unknowns. One station has been chosen forward of the

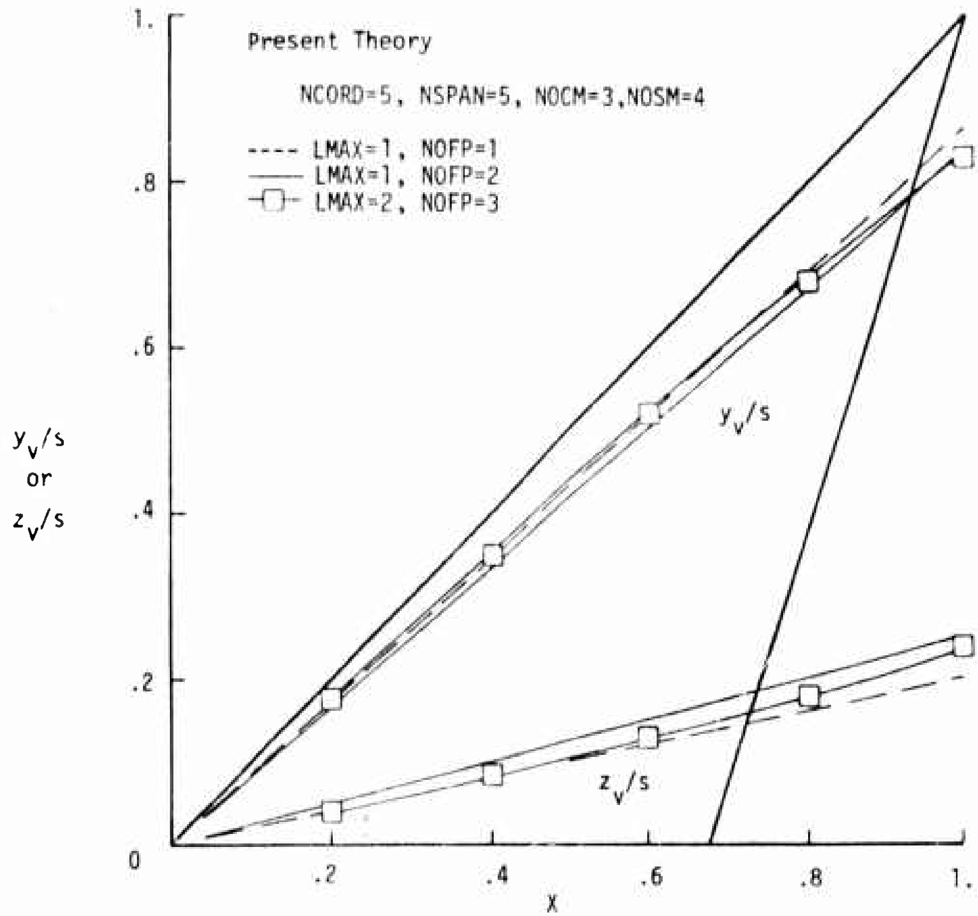


Figure 25. Leading-edge vortex position over right half of arrow wing (AR=2.02, λ = 71.2^o, α = 15.8^o).

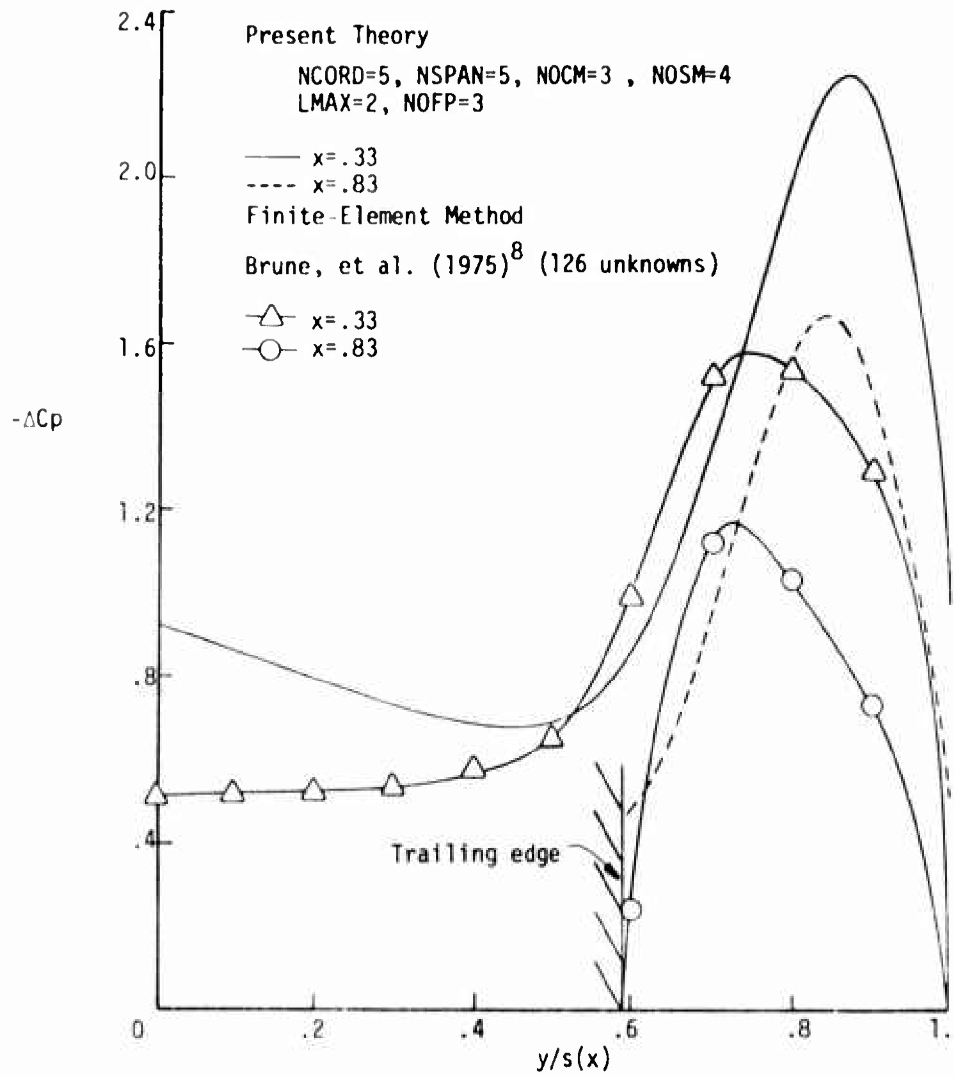


Figure 26. Comparison of theoretical loading models on arrow wing (AR=2.02, λ=71.2°, α = 15.8°).

root chord and the other has been selected aft of the trailing edge intrusion to indicate the effect of the Kutta condition at the trailing edge. Again, the different theories predict similar pressure distributions at these two stations. However, the present theory appears to retain the limitations of the Brown and Michael vortex-cut approximation in that the pressure peaks are higher and further outboard than those predicted by the lifting surface theory program of Brune, et al. (1975)[8] which utilizes a vortex-sheet representation. Also, the present loading predictions satisfy a modified Kutta condition at both the trailing and leading edges, and the pressure is not required to vanish at these points. This does not appear to be a serious problem, since the differences in the pressure distributions from zero contribute only slightly to the total loading on the wing due to the relatively large slopes in the pressure distributions near the wing edges.

This concludes the section on the revised vorticity modes. Preliminary results are promising, but limitations in the present procedure remain.

7. Conclusions and Recommendations

In conclusion, a lifting surface program based on the kernel function procedure has been developed to include leading-edge vortices. The present scheme can be generalized to arbitrary planforms and to include arbitrary sources of free vortices, but its use will probably be restricted by the computational effort.

With the present computer programs, results were first obtained for the delta wing of unit aspect ratio. Comparison with experiments indicate reasonable predictions of the loading with the inherent limitations that a Brown and Michael vortex-cut model imposes. It has been illustrated in the Introduction that a relatively small fraction of the vortex strength (less than 20 per cent) must be incorporated into the sheet to obtain the benefits of the Smith-type models for the slender-body problem.

Results were also obtained for the arrow wing to demonstrate the use of the program for more general planforms. These results emphasized the importance of a better representation of the Kutta condition at the trailing edge for such reentrant surfaces, than was originally employed. A simple bound vorticity model was first used to represent the vorticity feeding the leading-edge vortices, and this vorticity was discontinuously turned parallel to the free stream direction at the trailing edge. Convergence difficulties were encountered as one increased the sweep angle of the trailing edge, and consequently, an alternative bound vorticity distribution was developed to provide a smooth satisfaction of the linear Kutta condition at the trailing edge. Due to time limitations, only a few

runs were made with this revised model, but better convergence has been obtained for the arrow wing case at least. Furthermore, in deriving these new vorticity modes, a general procedure was developed which should provide bound vorticity modes for arbitrary planforms.

In general, the feasibility of the procedure has been demonstrated. Furthermore, an indication of the cause of previous convergence difficulties with programs which had attempted to satisfy the downwash and no-force conditions sequentially was presented, using the simpler slender-body representations. These results suggest that it is necessary to satisfy the boundary conditions simultaneously to obtain convergence.

Much work remains to be done to improve the usefulness of the present lifting surface program. First, it would be advantageous to further reduce the computational effort required to calculate the velocity contributions from the bound vorticity which feeds the leading-edge vortex. Secondly, it seems that a more accurate prediction of the loading and the vortex position can be obtained by a more complete representation of the leading-edge vortex sheet. This would entail additional degrees of freedom in the orientation of the vorticity leaving at the leading edge. An additional no-force boundary condition on these elements would have to be imposed to determine their orientation. For some purposes, the present vortex-cut model may be adequate, if the results are used in conjunction with slender-body theory corrections. For example, one can use the present procedure to calculate the leading-edge vortex location with the limitation that although the vertical position will be accurate, the spanwise position will, in reality, be further inboard.

Another field of interest would be the application of the lifting surface program to wings of higher aspect ratios. Recently, Nathman, Norton, and Rao(1976)[20] have published pressure distributions for less slender delta wings with aspect ratios of three and four and for some related double-delta planforms. One difficulty with such wings is that vortex bursting occurs over the wing at lower angles of attack as the apex angle of the delta wing is increased. Also, at higher angles of attack, the vortex core is not well defined and is replaced by a turbulent core of vorticity.

Additional effort may still be required to model the no-load condition on the trailing vortex sheet. Presently, only the linear, but not the nonlinear, no-load condition is being satisfied on the wake. This does not appear to be too serious in light of the results of Brune, et al. (1975)[8] and Kandil, et al. (1974)[7], which indicate that this is a fair representation of the wake. Finally, additional work still needs to be done to develop the new set of vorticity modes presented in this report for other planforms. Questions of resolution and convergence for this modal method remain to be answered, although significant progress has been made for the delta and arrow wing planforms.

The above extensions have been suggested by the present investigation, and their successful implementation would greatly enhance the versatility of this three-dimensional lifting surface program which includes leading-edge vortices.

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APPENDIX A

Evaluation of Upwash Integrals

The second and third integrals in Equation 13 will be evaluated explicitly here for the arrow wing configuration. For arbitrary configurations, one integration will be easy to perform, but the second integration may then have to be performed numerically. This should not present any difficulty, as the singularity will appear as a Cauchy Principal Value, which can be handled by a variety of techniques.

The second integral in Equation 13 becomes

$$B \equiv \frac{1}{4\pi} \oint\oint_{Sw} \frac{(x'-x)\, dx'dy'}{[(x-x')^2+(y-y')^2]^{3/2}}$$

$$= \frac{1}{4\pi} \oint_{-s}^{s} \oint_{\frac{y'}{s}}^{y'/s(1-c_r)+c_r} \frac{(x'-x)\, dx'\, dy'}{[(x-x')^2 + (y-y')^2]^{3/2}} \qquad \text{(A.1)}$$

where s is the semispan and c_r is the root chord, nondimensionalized by the chordwise length. Carrying out the integration and retaining only the finite part of the integral yields

$$B = \frac{1}{4\pi} \left\{ \frac{s}{\sqrt{s^2 + (1-c_r)^2}} \left[\sinh^{-1} \frac{(1-c_r)\,(c_r-x) + ys}{|\, y(1-c_r) - s(c_r-x)\,|} \right.\right.$$

$$- \sinh^{-1} \frac{(1-c_r)(1-x) + s(s+y)}{|y(1-c_r) - s(c_r-x)|} - \sinh^{-1} \frac{(1-c_r)(1-x) + s(s-y)}{y(1-c_r) + s(c_r-x)}$$

$$+ \sinh^{-1} \frac{(1-c_r)(c_r-x) - ys}{y(1-c_r) + s(c_r-x)} \Bigg] + \frac{s}{\sqrt{s^2+1}} \Bigg[\sinh^{-1} \frac{x-sy}{y+sx}$$

$$+ \sinh^{-1} \frac{s(s+y) + (1-x)}{y + sx} + \sinh^{-1} \frac{s(s-y) + (1-x)}{sx-y} + \sinh^{-1} \frac{x+ys}{sx-y} \Bigg] \Bigg\} \tag{A.2}$$

The remaining integral can be evaluated similarly.

$$C = + \frac{1}{4\pi} \iint_{Sw} \frac{(y-y')\, dx'\, dy'}{[(x-x')^2 + (y-y')^2]^{3/2}} \tag{A.3}$$

$$C = - \frac{1}{4\pi} \Bigg\{ \frac{1-c_r}{\sqrt{(1-c_r)^2 + s^2}} \Bigg[\sinh^{-1} \frac{sy + (1-c_r)(c_r-x)}{|-y(1-c_r) + s(c_r-x)|}$$

$$- \sinh^{-1} \frac{s(s+y) + (1-c_r)(1-x)}{|-y(1-c_r) + s(c_r-x)|} + \sinh^{-1} \frac{s(s-y) + (1-c_r)(1-x)}{y(1-c_r) + s(c_r-x)}$$

$$- \sinh^{-1} \frac{(1-c_r)(c_r-x) - sy}{y(1-c_r) + s(c_r-x)} \Bigg] + \frac{1}{\sqrt{1+s^2}} \Bigg[\sinh^{-1} \frac{x-sy}{y+sx}$$

$$+ \sinh^{-1} \frac{(1-x) + s(s+y)}{y + sx} - \sinh^{-1} \frac{(1-x) + s(s-y)}{sx - y} - \sinh^{-1} \frac{sy + x}{sx-y} \Bigg] \Bigg\} \tag{A.4}$$

These results reduce to those for the delta wing case when $c_r = 1$, and are then equivalent with results obtained by Nangia and Hancock (1968)[10], within a few sign errors which appear in their report.

APPENDIX B

Newton's Method for the Slender-Body Problem

This appendix expands the description of the use of Newton's method for the slender-body problem provided in the section on the Numerical Procedure. Originally, Newton's method was used solely to determine the vortex location. Later, it was employed to determine the vorticity distribution on the wing as well.

The flat plate delta wing problem under the restrictions of slender-body theory and conical flow was solved by Brown and Michael (1955)[4]. This problem can be formulated in the complex plane, $\omega = y + iz$ (see Figure B.1) as the complex potential W, due to a flat plate perpendicular to the flow and a pair of vortices.

$$W(\omega) = \frac{-i\Gamma}{2\pi} \ln \frac{\sqrt{\omega^2-1} - \sqrt{\omega_1{}^2-1}}{\sqrt{\omega^2-1} + \sqrt{\bar{\omega}_1{}^2-1}} - i\alpha\sqrt{\omega^2-1} \tag{B.1}$$

where all quantities have been nondimensionalized. ω_1 represents the complex vortex location, $\omega_1 = y_v + iz_v$, and $\bar{\omega}_1$ represents the complex conjugate of ω_1. Γ is the vortex strength and α is the angle of attack. The Kutta condition of finite velocity at the leading edge can be written as

$$\frac{2\pi\alpha}{\Gamma} = \frac{1}{\sqrt{\omega_1{}^2-1}} + \frac{1}{\sqrt{\bar{\omega}_1{}^2-1}} \tag{B.2}$$

This equation can be used to calculate the circulation strength, Γ, in terms of the vortex location. The forces on the vortex-cut combination in the complex plane are

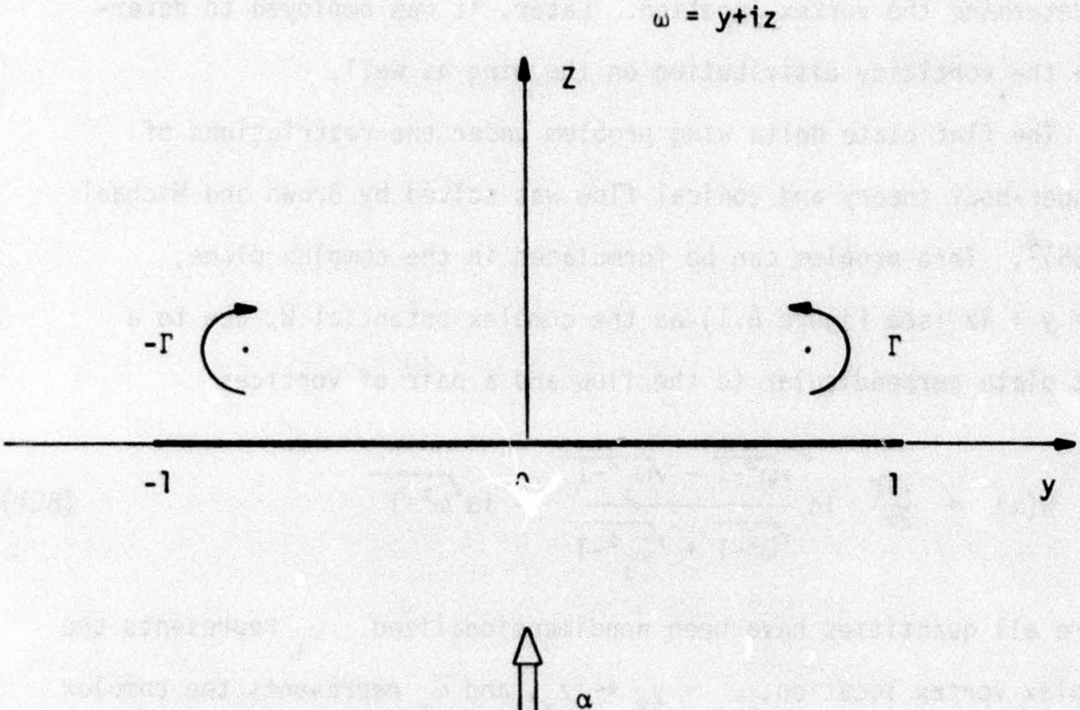

Figure B.1. Coordinate system for Brown and Michael slender-body delta wing problem.

$$F \equiv iF_y + F_z = -\lim_{\omega \to \omega_1} \left\{ \frac{dW}{d\omega} - \frac{\Gamma}{2\pi i} \frac{1}{\omega-\omega_1} \right\} + 2\bar{\omega}_1 - 1 = 0 \qquad (B.3)$$

The details of these derivations can be found in the original paper by Brown and Michael (1955)[4].

Since the potential specified in Equation B.1 automatically satisfies the downwash condition on the wing, the no-force condition (Equation B.3) provides the two real equations needed to determine the complex vortex position, $\omega_1 = y_v + iz_v$. Unfortunately, Equation B.3 is nonlinear in the vortex position variables; so a Newton's procedure was developed by Pullin (1973)[6] to solve this problem. A Newton's method is based on a linear extrapolation from some initial approximate solution and can be written in the following manner for this problem.

$$\begin{bmatrix} \Delta y_v \\ \Delta z_v \end{bmatrix} = \begin{bmatrix} \frac{\partial F_y}{\partial y_v} & \frac{\partial F_y}{\partial z_v} \\ \frac{\partial F_z}{\partial y_v} & \frac{\partial F_z}{\partial z_v} \end{bmatrix}^{-1} \begin{bmatrix} -F_y \\ -F_z \end{bmatrix} \qquad (B.4)$$

This equation gives an automatic procedure for obtaining an improved solution for the vortex location, if the residues, F_y and F_z, and the Jacobian matrix from the previous iteration are provided. The new vortex location is obtained from

$$y_v \text{ (new)} = y_v\text{(old)} + \Delta y_v$$

$$z_v \text{ (new)} = z_v\text{(old)} + \Delta z_v \qquad (B.5)$$

The derivatives for Equation B.4 can be obtained from

$$\frac{\partial F_y}{\partial y_v} = \text{Imag}\left[\frac{\partial F}{\partial y_v}\right]$$

$$\frac{\partial F_z}{\partial y_v} = \text{Real}\left[\frac{\partial F}{\partial y_v}\right]$$

$$\frac{\partial F_y}{\partial z_v} = \text{Imag}\left[\frac{\partial F}{\partial z_v}\right]$$

$$\frac{\partial F_z}{\partial z_v} = \text{Real}\left[\frac{\partial F}{\partial z_v}\right] \tag{B.6}$$

where

$$\frac{\partial F}{\partial y_v} = \frac{\partial F}{\partial \omega_1} + \frac{\partial F}{\partial \bar{\omega}_1}$$

$$\frac{\partial F}{\partial z_v} = i\left(\frac{\partial F}{\partial \omega_1} - \frac{\partial F}{\partial \bar{\omega}_1}\right)$$

This procedure provides convergence to the stable configuration in approximately three iterations if the initial approximation is within 10 per cent of the semispan of the final position. Unfortunately, the three-dimensional problem is more complicated than this, and some unexplained difficulties were encountered when a Newton's procedure was developed to find the vortex location in the lifting surface problem. In an effort to determine the cause of the difficulties, the slender-body problem was developed in a manner analogous to the three dimensional one.

The problem was formulated in the physical y,z plane and the wing was replaced by its bound vorticity representation. The downwash condition was no longer automatically satisfied and could be written as

$$\frac{1}{2\pi} \oint_{-1}^{1} \frac{\delta(y')\,dy'}{y - y'} + \alpha + \frac{\Gamma}{2\pi} \text{Real}\left[\frac{1}{y-\omega_1} - \frac{1}{y+\bar{\omega}_1}\right] = 0 \qquad (B.7)$$

This equation can be rearranged to yield

$$-\frac{1}{2\pi} \oint_{-1}^{1} \frac{\delta(y')\,dy'}{y - y'} = \frac{\Gamma}{2\pi} \text{Real}\left[\frac{1}{y-\omega_1} - \frac{1}{y+\bar{\omega}_1}\right] + \alpha \equiv f(y) \qquad (B.8)$$

This was inverted analytically to provide a check for the numerical procedure being developed. The inversion of Equation B.8 yields

$$\delta(y) = \frac{2}{\pi} \sqrt{1-y^2} \oint_{-1}^{1} \frac{f(y')}{y - y'} \frac{1}{\sqrt{1-y'^2}}\, dy' \qquad (B.9)$$

where δ (y) is the vorticity distribution on the wing and is equivalent to the difference in the spanwise velocity on the upper and lower surfaces.

For the choice of loading modes,

$$\delta(y) = \sum_{n=1}^{N} a_n \sqrt{1-y^2}\; y^{2n-1} \qquad (B.10)$$

the integral in Equation B.7 can be done analytically and the unknown a_n's can be obtained from a simple matrix inversion by chosing more collocation points at which the downwash condition is satisfied on the wing than the number of unknown modal coefficients, once a vortex position has been assumed.

Now the Kutta condition at the leading edge is automatically satisfied by the loading functions. The forces on the vertex become

$$iF_y + F_z = i\alpha + \frac{i\Gamma}{2\pi}\left\{\int_{-1}^{1} \frac{\delta/\Gamma dy'}{\omega_1 - y'} - \frac{1}{\omega_1 + \bar{\omega}_1}\right\} + 2\bar{\omega}_1 - 1 \qquad \text{(B.11)}$$

Originally, an attempt was made to satisfy the downwash condition (Equation B.7) and the no-force condition (Equation B.11) sequentially in the manner of Nangia and Hancock (1968)[10]. An initial vortex position was assumed and the downwash condition was applied at enough points to find an initial vorticity distribution provided by the a_n's and Γ. This distribution was then introduced into Equation B.11 which could be used in conjunction with Equation B.4 to obtain a better approximation for the vortex position. After the no-force condition was satisfied by moving the vortices, the downwash condition was no longer satisfied. Thus, the procedure sequentially updated the vorticity coefficients and the vortex position in an effort to satisfy both the downwash and the no-force conditions. However, as mentioned in the section on the Numberical Procedure, this scheme failed to converge as the procedure oscillated between the true solution and a false solution where the forces vanished, but where the downwash condition was not satisfied. As a result, convergence was not obtained.

Therefore, the decision was made to attempt to satisfy the downwash condition and the no-force condition simultaneously by a Newton's procedure. One obtains the following iteration scheme to update the initial approximation.

$$\begin{bmatrix} \Delta A \\ \Delta \Gamma \\ \Delta y_v \\ \Delta z_v \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial A} & \frac{\partial w}{\partial \Gamma} & \frac{\partial w}{\partial y_v} & \frac{\partial w}{\partial z_v} \\ \frac{\partial F}{\partial A} & \frac{\partial F}{\partial \Gamma} & \frac{\partial F}{\partial y_v} & \frac{\partial F}{\partial z_v} \end{bmatrix}^{-1} \begin{bmatrix} -W \\ -F \end{bmatrix} \tag{B.12}$$

where

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_{n+1} \end{bmatrix}$$

$$F = \begin{bmatrix} F_y \\ F_z \end{bmatrix}$$

This scheme resulted in convergence for the sample case being considered of the unit aspect ratio delta wing in approximately eight iterations for an initial location of the vortex within 10 per cent of the semispan of the final solution. Details of the rate of convergence for this problem were presented in Figure 12.

Due to the success of this procedure in the slender-body problem, a Newton's method has been developed for the lifting surface problem. However, success is not guaranteed as the three-dimensional problem involves a great many more variables than the slender-body problem. This additional complexity will result in slower convergence rates and may cause additional difficulties as well.

APPENDIX C

Listing of FORTRAN Programs

This Appendix consists of the primary programs described in the report. The listings are documented by comment cards. For additional details, see the section titled "Program Description" in this report.

All coding is in FORTRAN IV and the programs were run on the IBM 370/168 at M.I.T. Approximately 20 iterations of Program V can be performed in one minute of CPU time for the following choice of parameters: three no-force points (NOFP =3), two degrees of freedom in the vortex position (LMAX = 2), five chordwise and five spanwise collocation stations (NCORD = 5 , NSPAN = 5), four chordwise modes (NOCM = 4), and five spanwise modes (NOSM = 5). The choice of these parameters should be dictated by adequate resolution in the final answer.

Duplicate subroutines have not been listed. Duplicate subroutines are generally listed with Program V. The exceptions are the function subprograms B and XLE for Program IIIA which are listed with Program I.

C.1. Program I

The following listing for Program I includes Program I and the subprograms ASINH, A2, A4, B, B1, B2, B6, IGWW, and XINTGR.

Program I calculates the downwash coefficients due to the bound vorticity which feeds the leading-edge vortices. The output from Program 2 is used by Program IIIA to calculate the initial approximation for the vorticity distribution and is used by Program V to calculate the downwash residue on the wing.

```
C                   PROGRAM I
C
C CALCULATES DOWNWASH COEFFICIENTS DUE TO BOUND VORTICITY WHICH FEEDS
C      LEADING-EDGE VORTICES
C
C         INPUT  NCORD,NSPAN,S,CR        2I10   2F10.4
C         INPUT  NOCM,J1        2I10
C
C         PRIMARY OUTPUT   GWW       5F14.5
C
C   NEED FUNCTIONS   A1,A2,A4,A5,    B1,B2,B5,B6,GVORT,ASINH,B,XLE
C   NEED SUBROUTINES   BLOCK DATA,COLPT,IGWW ,XINTGR
C
      DIMENSION XPT(5),YPT(5),COEFF(25,5),
     CSGW1(5),SGW2(5),SGW3(5),SGW4(5),SGW5(5) ,SGW6(5)
      COMMON XPI,YPJ,S,M,MP/PLAN/CR    /GAUS/G(24),W(24)/MODES/NOCM
      EXTERNAL A1,A2,A4,A5,B1,B2,B5,A6,B6
C
C INITIALIZE VARIABLES
       DATA COEFF/125*0./
      DO 150 JDUMMY = 13,24
      W(JDUMMY)=W(25 -JDUMMY)
  150 G(JDUMMY) = -G(25-JDUMMY)
      PI=3.141593
      WRITE(6,910)
C
      READ(5,920) NCORD,NSPAN,S,CR
C
C  NCORD = NO. OF CHORDWISE COLLOCATION POINTS
C  NSPAN = NO. OF SPANWISE COLLOCATION POINTS
C  S = SEMISPAN; NON-D BY MAXIMUM LENGTH
C  CR = ROOT CHORD; NON-D BY MAXIMUM LENGTH
C
      READ(5,920)     NOCM,J1
C
C  NOCM = NO. OF CHORDWISE MODES


C  J1 IS CONTROL PARAMETER: IF J1=1, NCORD2=NCORD; ELSE NCORD2=NCORD+1
      WRITE(6,970) NCORD,NSPAN,S,NOCM,CR
C
C CALCULATE LOCATION OF COLLOCATION POINTS
      CALL COLPT(NCORD,NSPAN,XPT,YPT)
       IF(J1.EQ.1) GO TO 300
      NCORD2=NCORD+1
      XPT(NCORD2)=(XPT(NCORD)+XPT(NCORD-1))/2.
  300 CONTINUE
      IF(J1.EQ.1) NCORD2=NCORD
       WRITE(6,930)
C
C DEFINE LIMITS OF INTEGRATION IN SPANWISE DIRECTION
C        C'S REFER TO LEFT-HAND SIDE OF RESPECTIVE REGION
C        D'S REFER TO RIGHT-HAND SIDE OF RESPECTIVE REGION
      C1 =0.
      D5=S
      C6 = -S
      D6 = 0.
C
C CALCULATE COEFFICIENTS AT EACH COLLOCATION POINT
      DO 400 I=1,NCORD2
      DO 400 J=1,NSPAN
       NI=J+(I-1)*NSPAN
      XPI=XLE(YPT(J))+B(YPT(J))*XPT(I)
      YPJ=YPT(J)*S
C
C OUTPUT LOCATION OF COLLOCATION POINTS
      WRITE(6,940) NI,XPT(I),YPT(J),XPI
      ETA=.02
      IF((1.-XPI).LT..02) ETA=1.-XPI
      D1=S*(XPI-.02)
      C2 = 0.
      IF((XPI-.02).GT.CR) C2 = S*((XPI-.02)-CR)/(1.-CR)
      D2 = YPJ-.02
      C3=YPJ-.02
```

```
                  FUNCTION ASINH(Z)
C
C  ASINH(Z) CALCULATES INVERSE HYPERBOLIC SINE
C
      IF(Z.LT.-10.) GO TO 20
       ASINH = ALOG(Z+SQRT(1.+Z*Z))
      RETURN
C USE EXPANSION FORM FOR ASINH FOR LARGE NEGATIVE VALUES OF Z
   20 ASINH = ALOG((1./(4.*Z*Z)-1.)/Z) -.693147
      RETURN
      END
```

```
                  FUNCTION A2(Y)
C
C  A2(Y) PROVIDES INITIAL INTEGRATION POINT IN X DIRECTION IN REGION 2
C                ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C                Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N
      A2 = XPT-.02
      RETURN
      END
C
C**************************************************************************
C
                  FUNCTION A4(Y)
C
C  A4(Y) PROVIDES INITIAL INTEGRATION POINT IN X DIRECTION IN REGION 4
C                ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C                Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N
      IF (Y-S*(XPT-.02)) 10,10,20
   10 A4 = XPT-.02
      RETURN
   20 A4 = Y/S
      RETURN
      END
```

```
                    FUNCTION B(S)
C
C   B(S) CALCULATES LOCAL CHORD; NON-D BY MAXIMUM LENGTH
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              S:    SPANWISE COORDINATE; NON-D BY SEMISPAN
C
      COMMON /PLAN/ CR
      B = CR*(1.-S)
      RETURN
      END
C
C*****************************************************************************
C
                    FUNCTION XLE(S)
C
C   XLE(S) CALCULATES LOCATION OF LEADING EDGE; NON-D BY MAXIMUM LENGTH
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              S:    SPANWISE COORDINATE; NON-D BY SEMISPAN
C
      XLE = S
      RETURN
      END
```

```
                    FUNCTION B1(Y)
C
C   B1(Y) PROVIDES FINAL INTEGRATION POINT IN X DIRECTION IN REGION 1
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
       COMMON XPT,YPT,S,M,N /PLAN/CR
       B = Y*(1.-CR)/S + CR
      IF (B.GT.(XPT-.02)) B=XPT-.02
      B1 = B
       RETURN
      END
C
C*****************************************************************************
C
                    FUNCTION B2(Y)
C
C   B2(Y) PROVIDES FINAL INTEGRATION POINT IN X DIRECTION IN REGION 2
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
       COMMON XPT,YPT,S,M,N /PLAN/CR,ETA
       B = Y*(1.-CR)/S + CR
      IF(B.GT.(XPT+.02)) B = XPT+.02
      IF (B.GT.1.) B=1.
      B2 = B
      RETURN
      END
C
```

```
C*****************************************************************************
C
                  FUNCTION B6(Y)
C
C  B6(Y) PROVIDES FINAL INTEGRATION POINT IN X DIRECTION IN REGION 6
C             ARROW WING CONFIGURATION
C
C        ARGUMENT LIST
C             Y:   SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
       COMMON XPT,YPT,S,M,N /PLAN/CR
      B6 = -Y*(1.-CR)/S + CR
      RETURN
      END
```

```
                  SUBROUTINE  IGWW(C,D,A,B,SGWW)
C
C  IGWW  CALUCLATES DOWNWASH INTEGRAL
C
C        ARGUMENT LIST
C             C:   LOWER LIMIT OF INTEGRAL
C             D:   UPPER LIMIT OF INTEGRAL
C             A:   FUNCTION DESCRIBING LOWER LIMIT OF INTEGRAL
C             B:   FUNCTION DESCRIBING UPPER LIMIT OF INTEGRAL
C             SGWW:  INTEGRALS
C
      COMMON XPT,YPT,S,MDUM,MPDUM
      COMMON/GAUS/G(24),W(24)/MODES/NOCM
      DIMENSION          SGWW(5),ENTGD(  5),SUM(  5),GVORS(5)
C
      ENTG4(X,Y)=(XDIFF*(X*GVOR2-XPT*GVOR1     )+
     CYDIFF*(-Y*GVOR2+YPT*GVOR1    ))/ATHIRD
C
      PI=3.141593
      Z=1.E-9
C
C INITIALIZE SUMMATIONS
      DO 10 MQ=1,NOCM
      M=MQ-1
      ENTGD(MQ)=0.0
      SUM(MQ)=0.0
      GVORS(MQ)=GVORT(M,XPT,YPT,S)
   10 CONTINUE
C
C DO SPANWISE INTEGRAL
      DO 200 J=1,24
      Y=((D-C)*G(J)+D+C)/2.
      BY=B(Y)
      AY=A(Y)
      AP=BY-AY
      BP=BY+AY
```

```
C
C DO CHORDWISE INTEGRAL
      DO 100 I=1,24
      X=(AP*G(I)+BP)/2.
      XDIFF=X-XPT
      YDIFF=YPT-Y
      ATHIRD=XDIFF*XDIFF+YDIFF*YDIFF+Z*Z
      ATHIRD=ATHIRD*SQRT(ATHIRD)
      DO 400 MQ=1,NOCM
      M=MQ-1
      GVOR1=GVORS(MQ)
      GVOR2=GVORT(M,X,Y,S)
      ENTGD(       MQ)=ENTGD(       MQ)+ENTG4(X,Y)*W(I)
  400 CONTINUE
  100 CONTINUE
      DO 300 MQ=1,NOCM
      SUM(     MQ)=SUM(     MQ)+ENTGD(     MQ)*W(J)*AP
      ENTGD(     MQ)=0.0
  300 CONTINUE
  200 CONTINUE
      CONST=(D-C)/(16.*PI)
      DO 500 MQ=1,NOCM
      SGWW(MQ)=CONST*SUM(       MQ)
  500 CONTINUE
      RETURN
      END
```

```
                  SUBROUTINE XINTGR(ENTG2,ENTG3)
C
C  XINTGR CALCULATES THE 'SINGULAR' CONTRIBUTION TO THE
C        DOWNWASH INTEGRAL
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              ENTG2: SPANWISE VORTICITY COMPONENT
C              ENTG3: CHORDWISE VORTICITY COMPONENT
C
       COMMON XPT,YPT,S,M,N /PLAN/CR
      FACT1 = ABS(S*(CR-XPT) -YPT*(1.-CR))
      FACT2 = S*(CR-XPT) +YPT*(1.-CR)
      ISING = 1
      IF (ABS(YPT*(1.-CR) - S*(CR-XPT)).LE..0001) ISING = 0
      IF (ISING.EQ.0) TERM1 = ALOG((S+YPT)/YPT)
       IF (ISING.EQ.0) GO TO 200
      TERM1 = ASINH ((S*(S+YPT) + (1.-CR)*(1.-XPT))/FACT1)
     C -ASINH((S*YPT + (CR-XPT)*(1.-CR))/FACT1)
  200 CONTINUE
      TERM2 = ASINH((S*(S-YPT) + (1.-CR)*(1.-XPT))/FACT2)
     C -ASINH((-S*YPT + (1.-CR) *(CR-XPT))/FACT2)
      TERM3 = ASINH((S*(YPT+S)+ (1.-XPT))/(S*XPT+YPT))
     C +ASINH((XPT-YPT*S)/(S*XPT+ YPT))
      TERM4 = ASINH((S*(S-YPT) + (1.-XPT)) /(S*XPT-YPT))
     C +ASINH((XPT+ S*YPT) /(S*XPT-YPT))
       FACT1 = SQRT(S*S+(1.-CR)**2)
      FACT2 = SQRT(1.+S*S)
      ENTG2 = -S/FACT1*(TERM1+TERM2)  +S/FACT2*(TERM3+TERM4)
      ENTG3 = -((CR-1.)/FACT1*(TERM1-TERM2)+ 1./FACT2*(TERM3-TERM4))
      RETURN
      END
```

C.2. Program WOW

The following listing for Program WOW includes Program WOW and the subprograms CHDWS, KERNL, MNGLR, and PLOT.

Program WOW calculates the downwash coefficients due to the horseshoe vortices on the wing. The output from Program WOW is used by Program IIIA to calculate the initial approximation for the vorticity distribution and is used by Program V to calculate the downwash residue on the wing.

```
C                    PROGRAM WOW
C
C CALCULATES COEFFICIENTS OF CHOSEN MODES OF THE VORTICITY DISTRIBUTION
C      AS A NUMERICAL SOLUTION OF THE INTEGRAL EQUATION RELATING
C      THE STRENGTH OF HORSESHOE VORTICES AND DOWNWASH
C
C        INPUT   NOST,NI(3),SPAN        2I5  F10.4
C        INPUT   NOCP,NOLT,NCP,MP,J1,J2,CSR        4I5,2I5,F14.5
C        INPUT   XOC,SOS,ETA,NI(2),NI(4)     3F10.4  3I2
C
C        PRIMARY OUTPUT  DWR  5E14.5
C
C   NEED SUBPROGRAMS  B,XLE,CHDWS,FUNCTN,KERNL,MNGLR,PLOT,BLOCK DATA
C
      DIMENSION NI(4),XVECT(25),YVECT(25),S(4),W(4),DWR(25,25),F(5)
      COMMON AR(4,5,5),ALS(5,10),CR(5),TKR(10),XOC,SOS,Y,Z,
     C YMN,ZMZ,RSQR,ETA,GAUSX(10),PO2,NOLT,NCP,MP,N,X,GZ,
     C J1,J2,GS,YMN2,ZMZ2,CSR  /PLAN/CXMAX
      COMMON                  /GAUN/CN(10,10),WN(10,10) /MODES/NOST,NINC
C
C INITIALIZE VARIABLES
      DATA S(3),S(4),W(3)/-1.0,0.,1.0/
      PO2=1.5707963
      NINC=-1
C
       READ 100,    NOST,NI(3),SPAN
C
C  NOST =NO. OF SPANWISE LOADING MODES
C  NI(J) =NO. OF LEGENDRE-GAUSS POINTS IN SPANWISE INTEGRATION
C        IN REGION J
C  SPAN =SPAN
       JS=4
C
C   JS=NO. OF INTEGRATION REGIONS IN SPANWISE DIRECTION
C
    5  READ 501,    NOCP,NOLT,NCP,MP,J1,J2,      CSR

C
C  NOCP =NO. OF COLLOCATION POINTS IN HALF WING
C  NOLT =NO. OF CHORDWISE LOADING MODES
C  MP =NO. OF CHORDWISE LEGENDRE-GAUSS QUADRATURE POINTS IN MNGLR
C  NCP =NO. OF CHORDWISE LEGENDRE-GAUSS QUADRATURE POINTS IN CHDWS
C  J1,J2  CONTROL OUTPUT; NORMALLY 0
C  CSR =CHORD TO SPAN RATIO
      CXMAX = CSR*SPAN
       WRITE(6,930)
      NMODE=NOST*NOLT
      WRITE(6,911) SPAN,   NOLT,NOST,NOCP,CSR
      WRITE(6,900) NI(3),NCP,MP
C
C CALCULATE COEFFICIENTS AT THE COLLOCATION POINTS
       DO 90  L=1,NOCP
C
       READ 6,XOC,SOS,ETA,N,NI(2),NI(4)
C
C  XOC =FRACTION OF CHORD; 0 AT LE, 1.0 AT TE
C  SOS =FRACTION OF SEMISPAN; 0 AT ROOT, 1.0 AT TIP
C  ETA =ZETA=SMALL INTEGRATION REGION ABOUT SINGULARITY
C  N =SECTION NO. INDICATOR
       X=XLE(N,SOS)+2.*B(N,SOS)*XOC
C
C  XLE =LOCATION OF LEADING EDGE; REF: ROOT SEMICHORD
C  B =LENGTH OF LOCAL SEMICHORD; REF:ROOT SEMICHORD
       Y=SOS
      XVECT(L) = X
      YVECT(L) = Y
      IF(SOS+ETA.LT.1.) GO TO 30
      ETA=1.-SOS
C NO REGION 2
      NI(2)=0
   30 CONTINUE
      IF(SOS-ETA.GT.0.0) GO TO 40
      ETA=SOS
```

```
C NO REGION 4
      NI(4)=0
  40  S(2)=SOS+ETA
C
C OUTPUT COLLOCATION POINT AND RELATED DATA
      WRITE(6,910) L,XOC,SOS,ETA,NI(2),NI(4),X
       W(2)=1.0-S(2)
       W(4)=SOS-ETA
C
C  S =LEFT-HAND LIMIT OF INTERVAL
C  W =LENGTH OF INTERVAL
C
C INITIALIZATION OF INTEGRALS FOR EACH INTEGRATION REGION
       DO 41  I=1,JS
       DO 41 N1=1,NOLT
       DO 41 N2=1,NOST
       AR(I,N1,N2)=0.0
  41   CONTINUE
C
C DO INTEGRALS IN REGIONS WITH NO SINGULARITY BY GAUSSIAN QUADRATURE
       DO 500 I=2,JS
 411   NSIP=NI(I)
C
C   NSIP=NO. OF INTEGRAL POINTS
      IF(NSIP.EQ.0) GO TO 500
       DO 50  J=1,NSIP
      GS=S(I)-(GN(J,NSIP)-1.0)/2.*W(I)
C
C  GN(J,NSIP) =JTH ABSCISSA OF LEGENDRE-GAUSS QUADRATURE OF ORDER NSIP
      GY=GS
       YMN=Y-GY
      YMN2=YMN*YMN
      RSQR=YMN2
      WT=WN(J,NSIP)*      W(I)/(2.*RSQR)
C
C  WN(J,NSIP) =JTH WTG. FUNCTION OF LEGENDRE-GAUSS QUADRATURE


C
C CALCULATE VORTICITY STRENGTH
      CALL FUNCTN(NOST,GS,F)
C
C DO CHORDWISE INTEGRATION
       CALL CHOWS
       DO 45 M=1,NOLT
       DO 45 NSF=1,NOST
       AR(I,M,NSF)=AR(I,M,NSF)+CR(M)*F(NSF)*WT
C
C  AR(I,M,NSF) = SURFACE INTEGRAL IN REGION I
  45   CONTINUE
  50   CONTINUE
 500    CONTINUE
C
C DO INTEGRAL OF MANGLER-TYPE SINGULARITY
       CALL MNGLR(NOST)
       DO 60  I=1,NMODE
       DWR(L,I)=0.0
  60   CONTINUE
C
C SUM INTEGRALS OVER ALL REGIONS OF INTEGRATION
       DO 70  I=1,NOLT
       DO 70  J=1,NOST
       K=I+NOLT*(J-1)
       DO 70  MS=1,JS
       DWR(L,K)=DWR(L,K)+AR(MS,I,J)
C
C  DWR =REAL PART OF GENERALIZED AERODYNAMIC INFLUENCE COEFFICIENTS
  70  CONTINUE
   90 CONTINUE
        NOCO=(NMODE+4)/5
C
C OUTPUT AERODYNAMIC INFLUENCE COEFFICIENTS
      DO 110 L=1,NOCP
      DO 110 K=1,NOCO
```

```
      JMAX=5*K
      JMIN=JMAX-4
      WRITE(6,81)  (DWR(L,JC),JC=JMIN,JMAX),L,K
      WRITE(7,920) (DWR(L,JC),JC=JMIN,JMAX),L,K
  110 CONTINUE
C
C PLOT PLANFORM AND COLLOCATION POINTS
      CALL PLOT(XVECT,YVECT,NOCP)
  6   FORMAT(3F10.4,3I2)
   81 FORMAT(5E15.7,2X,'DWR',I2,1X,I2)
  91  FORMAT(1H0,'SPAN=',F8.3,3X,                        'NO. OF CHOWS L
     COADING MODE=',I3,3X,'NO. OF SPWS LOADING MODE=',I4,/,1X,'NO. OF
     CCOLLOCATION PTS=',I4,3X,'CHORD TO SPAN RATIO=',F10.6,/)
 100     FORMAT(2I5,F10.4)
 501    FORMAT(        4I5,2I5, F14.5)
  900 FORMAT(1H ,'INTEGRATION PTS. IN REGION 3=',I5,5X,'IN CHOWS=',I5,
     C5X,'IN MNGLR=',I5,//)
  910 FORMAT(1H ,'COLLOCATION PT.',I4,3X,'XOC=',F7.4,3X,'SOS=',F7.4,3X,
     C'ETA=',F7.4,3X,'NI(2)=',I3,3X,'NI(4)=',I3,3X,'X=',F7.4)
  920 FORMAT(5E14.6,2X,'DWR',I2,1X,I2)
  930 FORMAT(' WOW:  CALCULATES INFLUENCE COEFFICIENTS FOR DOWNWASH ON W
     CING:  APRIL 27,1977',//)
       STOP
        END
```

```
                      SUBROUTINE CHOWS
C
C  CHOWS: EVALUATION OF CHROWISE INTEGRAL USING GAUSSIAN QUADRATURE
C
        DIMENSION BETA(10),THETA(10),GX(10),AL(5)
       COMMON AR(4,5,5),ALS(5,10),CR(5),TKR(10),XOC,SOS,Y,Z,
     C YMN,ZMZ,RSOR,ETA,GAUSX(10),PO2,NOLT,NCP,MP,N,X,GZ,J1,J2,GS,YMN2,
     C ZMZ2,CSR /GAUN/GN(10,10),WN(10,10) /MODES/NOST,NINC
C
       SEMICO=B(N,GS)
        ELE=XLE(N,GS)
C
C INITIALIZE SUMMATIONS
        DO 1  I=1,NOLT
        CR(I)=0.0
 1      CONTINUE
C
C IF(RSOR-.1)>0, THE INTEGRAL IS EVALUATED AS A SINGLE INTEGRAL
       IF (RSOR-0.1) 21,3,3
C
C  NINC INSURES THAT ALS IS ONLY CALCULATED ONCE
   3  IF (NINC) 4,4,7
 4      NINC=2
        DO 5 I=1,NCP
        BETA(I)=(1.-GN(I,NCP))*PO2
      GX(I)=-COS(BETA(I))
      DO 5 J=1,NOLT
      ALS(J,I)=SIN(BETA(I)*FLOAT(J))/FLOAT(2**(2*J))*4.
C
C  ALS(J,I) = LOADING FUNCTIONS. REF: ASHLEY AND LANDAHL
 5         CONTINUE
 7      DO 6 I=1,NCP
 6         GAUSX(I)=X-(ELE+SEMICO*(1.+GX(I)) )
        CALL KERNL
        WGHT=PO2
        DO 20 I=1,NCP
```

```
      CW=WN(I,NCP)*SIN(THETA(I))*WGHT
      DO 20 J=1,NOLT
      CR(J)=ALS(J,I)*CW*TKR(I)+CR(J)
 20     CONTINUE
       GO TO 50
C
C FOR RSQR-.1<0, THE CHORDWISE INTEGRAL IS COMPUTED BY TWO GAUSSIAN
C      QUADRATURES TO HANDLE THE FINITE JUMP IN KERNEL AT X-X1=Y-Y1=0
 21   IF(X-ELE) 3,3,220
 220   IF(X-(ELE+2.*SEMICD)) 22,3,3
  22  THBD=ARCOS((ELE+SEMICD-X)/SEMICD)
       K=-1
       WGHT=THBD/2.
       DO 23 I=1,NCP
       THETA(I)=(1.-GN(I,NCP))*THBD/2.
 23        GAUSX(I)=X-(ELE+SEMICD*(1.-COS(THETA(I))))
       GO TO 35
 24      WGHT=PO2-WGHT
       K=1
       DO 25 I=1,NCP
       THETA(I)=THBD+(1.-GN(I,NCP))*(PO2-THBD/2.)
  25  GAUSX(I)=X-(ELE+SEMICD*(1.0-COS(THETA(I))))
 35        CALL KERNL
       DO 40  I=1,NCP
      CW=WN(I,NCP)*SIN(THETA(I))*WGHT
      DO 40 J=1,NOLT
      AL(J)=SIN(THETA(I)*FLOAT(J))/FLOAT(2**(2*J)) *4.
      CR(J)=AL(J)*CW*TKR(I)+CR(J)
C
C  CR = CHORDWISE INTEGRAL
 40        CONTINUE
       IF(K) 24,50,50
   50 CONTINUE
 60        RETURN
       END




                    SUBROUTINE KERNL
C
C  KERNL:  EVALUATION OF KERNEL FUNCTION FROM STEADY,NON-PLANAR,
C       INCOMPRESSIBLE LIFTING SURFACE THEORY. REF: ASHLEY AND
C       LANDAHL, CH. 5
C
     COMMON AR(4,5,5),ALS(5,10),CR(5),TKR(10),XOC,SOS,Y,Z,
    C YMN,ZMZ,RSQR,ETA,GAUSX(10),PO2,NOLT,NCP,MP,N,X,CZ,
    C J1,J2,CS,YMN2,ZMZ2,CSR
C
C  GAUSX(I) = X-X1
C  YMN = Y-Y1
   5 DO 10 I=1,NCP
     XME=GAUSX(I)*CSR
     XME2=XME*XME
     R=SQRT(RSQR+XME2)
     G=1.0+XME/R
     TKR(I)=-G
C
C  TKR = REAL PART OF KERNEL
  10 CONTINUE
  15 RETURN
     END
```

```
      SUBROUTINE MNGLR(NOST)
C
C  MNGLR:  COMPUTES PRINCIPAL VALUE OF A MANGLER INTEGRAL WHICH
C       INVOLVES A SINGULARITY AT Y=Y1. REF: WATKINS, NASA TN R-48
C
C        ARGUMENT LIST
C             NOST:   NO. OF SPANWISE MODES
C
      DIMENSION D(6),SW(6),CRTE( 5),AL(5),F(5)
      COMMON AR(4,5,5),ALS(5,10),CR(5),TKR(10),XOC,SOS,Y,Z,
     C YMN,ZMZ,RSQR,ETA,GAUSX(10),PO2,NOLT,NCP,MP,N,X,GZ,
     C J1,J2,GS,YMN2,ZMZ2,CSR   /GAUN/GN(10,10),WN(10,10)
C
      DO 1 I=1,NOLT
      CRTE(I)=0.0
    1 CONTINUE
      SKR=-2.0
      DATA SW/13.,72.,495.,495.,72.,13./
C
C  SW =WEIGHTING COEFFICIENTS AT THE RESPECTIVE INTEGRATION POINTS
      D(1)=ETA
       D(2)=ETA*2./3.
       D(3)=ETA/3.
      D(4)=-D(3)
       D(5)=-D(2)
       D(6)=-D(1)
C
C  D(I) = LOCATION OF INTEGRATION STATIONS W.R.T. Y WITHIN INTERVAL
C  DO LOOP 50 COMPUTES F1 THROUGH F7, EXCEPT F4
       DO 50 J=1,6
       GS=SOS*D(J)
       GY=GS
      CALL FUNCTN(NOST,GS,F)
       YMN=Y-GY
      YMN2=YMN*YMN
      RSQR=YMN2

      CALL CHOWS
      DO 40 L=1,NOLT
      DO 40 K=1,NOST
       W=SW(J)
       AR(1,L,K)=CR(L)*F(K)*W*AR(1,L,K)
C  AR = REAL PART OF SURFACE INTEGRAL
  40   CONTINUE
   50  CONTINUE
      CALL FUNCTN(NOST,SOS,F)
      THMAX=ARCOS(1.0-XOC*2.0)
C
C  DO LOOP 100 COMPUTES F4 AT Y= Y1
      DO 100 I=1,MP
       THETA=(1.-GN(I,MP))/2.*THMAX
      CW=WN(I,MP)*SIN(THETA)*THMAX/2.0
C
C  CW = WEIGHTING TERM
C
C  AL(L) = THE TWO-D CHORDWISE LOADING FUNCTIONS
      DO 70 L=1,NOLT
      AL(L)=SIN(FLOAT(L)*THETA)/FLOAT(2**(2*L))*4.
      CRTE(L)=AL(L)*CW*SKR+CRTE(L)
   70 CONTINUE
C
C  CRTE = CHORDWISE INTEGRAL
 100  CONTINUE
 103 FCTR = 100.*ETA
      DO 105 L=1,NOLT
       DO 105 K=1,NOST
      AR(1,L,K)=(-1360.0*CRTE(L)*F(K)*AR(1,L,K))/FCTR
C
C  AR: FINAL VALUE OF THE SURFACE INTEGRAL
 105 CONTINUE
       RETURN
       END
```

```
      SUBROUTINE PLOT(XVECT,YVECT,NOCP)
C
C  PLOT PLOTS PLANFORM AND CONTROL POINTS
C
C        ARGUMENT LIST
C              XVECT: CHORDWISE COORDINATE; NON-D BY ROOT CHORD
C              YVECT: SPANWISE COORDINATE; NON-D BY SEMISPAN
C              NOCP:  NO. OF COLLOCATION POINTS
C
      DIMENSION XVECT(25),YVECT(25),          NX(25),NY(25)
      INTEGER*2 A,D,C,LINE(101)
      DATA A,D,C/'  ','**','XX'/
      WRITE(6,910)
      SCALE = 1. + XLE(1,1.)
      DO 30 L=1,NOCP
       X = XVECT(L) +1.
       NX(L) = INT(100.*X/SCALE+.1) + 1
      NY(L)=51-INT(50.*YVECT(L)+.1)
   30 CONTINUE
      S=1.
      DO 100 I=1,51
      DO 10 J=1,101
   10 LINE(J)=A
      ELE = XLE(1,S) + 1.
      CHD = B(1,S) *2.
      XTE=ELE+CHD
      NLE = INT(100.*ELE/SCALE + .1) + 1
       NTE = INT(100.*XTE/SCALE + .1) + 1
      IF(I.NE.1.AND.I.NE.51) GO TO 50
      DO 20 ISTEP=NLE,NTE
      LINE(ISTEP)=D
   20 CONTINUE
      GO TO 60
   50 CONTINUE
      LINE(NLE)=D
      LINE(NTE)=D



   60 CONTINUE
      DO 40 L=1,NOCP
      IF(NY(L).EQ.I) LINE(NX(L))=C
   40 CONTINUE
      WRITE(6,900) LINE
      S=S-.02
  100 CONTINUE
  900 FORMAT(1H ,101A1)
  910 FORMAT('1 PLOT OF PLANFORM AND CONTROL POINTS:  NOT TO SCALE',///)
      RETURN
      END
```

C.3. Program IIIA

The following listing for Program IIIA includes Program IIIA and subprograms XGWGMW, AXA, DETERM, and PRESS.

Program IIIA calculates the initial approximation for the vorticity distributed from the leading-edge vortex location and the outputs of Program I and Program WOW. The output of Program IIIA is used to provide the initial vorticity distribution for Program V.

```
C                   PROGRAM IIIA
C
C IIIA CALCULATES VORTICITY COEFFICIENTS FROM VORTEX LOCATION AND
C      OUTPUTS OF PROGRAMS WOW AND I
C
C  NOTE:  DOWNWASH POSITIVE FOR WIDNALL      UPWASH POSITIVE HERE
C
C        INPUT  J1,J2,J3,J4          4I5
C        INPUT  NCORD,NSPAN,S,NOCM,NOSM,CR        2I10,F10.4,2I10,F10.4
C        INPUT  AWW,GWW         5E14.5
C        INPUT  LMAX,ALFA     I5  F10.6
C        INPUT  GYVOR,GZVOR          5E14.5
C
C        PRIMARY OUTPUT   VORTICITY COEFFICIENTS:
C             FIRST NOCM*NOSM MODES ARE HORSESHOE VORTEX MODES;
C             REMAINING ARE LEADING-EDGE VORTEX MODES      5E14.5
C
C   NEED FUNCTIONS  B,FW,XGWT,XI,XGWGMW,XLE
C   NEED SUBROUTINES  AXA,COLPT,DETERM,DFNCT,FNCTN,GAUSID,PRESS
C
      EXTERNAL XGWGMW,XGWT
      DIMENSION XPI(5),YPI(5),VR(25),ATVR(25),PR(25,25),AWW(25,20),
     C GWW(25,5),COEFF(25,25)
      COMMON XPI,YPJ,S,M,MP/GVOR/GYVOR(5),GZVOR(5)/PLAN/CR /SEC/ZPT
      COMMON/MODES/NOCM,NOSM/CONTRL/J2,J3/VLOC/LMAX /GAUS/G(24),W(24)
C
      DO 100 JDUMMY =13,24
      W(JDUMMY) = W(25-JDUMMY)
  100 G(JDUMMY) = -G(25-JDUMMY)
      DATA COEFF/625*0.0/
      PI=3.141593
C ON WING Z = 0
       ZPT=0.
      WRITE(6,860)
C
      READ(5,950) J1,J2,J3,J4


C
C  J1 = CONTROL PARAMETER. IF J1=1,NCORD2=NCORD; ELSE, NCORD2=NCORD+1
C  J2,J3 ARE CONTROLS; J2=1 CALLS DETERM; J3 > 1 CALLS ITERATION PROCEDURE
C  J4 CONTROLS OUTPUT; IF J4.EQ.1, OUTPUTS INTERMEDIATE RESULTS
       WRITE(6,950) J1,J2,J3,J4
C
      READ(5,880) NCORD,NSPAN,S,NOCM,NOSM,CR
C
C  NCORD = NO. OF CHORDWISE COLLOCATION POINTS
C  NSPAN = NO. OF SPANWISE COLLOCATION POINTS
C  S = SEMISPAN
C  NOCM = NO. OF CHORDWISE MODES
C  NOSM = NO. OF SPANWISE MODES
C  CR = ROOT CHORD DIVIDED BY MAXIMUM LENGTH
      WRITE(6,870) NCORD,NSPAN,S,NOCM,NOSM,CR
      NCORD2=NCORD+1
      IF(J1.EQ.1) NCORD2=NCORD
      NOCP= NCORD2  *NSPAN
      NMOD=NOSM*NOCM
      NMODT=NMOD+NOCM
C
C  NOCP = NO. OF COLLOCATION POINTS
C  NMOD = NO. OF HORSESHOE VORTEX MODES
C  NMODT = TOTAL NUMBER OF MODES
      DO 200 I=1,NOCP
C
      READ (5,900) (AWW(I,J),J=1,NMOD)
C
C  AWW = DOWNWASH INFLUENCE COEFFICIENTS FROM PROGRAM WOW
       DO 200 J=1,NMOD
      COEFF(I,J)=-AWW(I,J)
  200 CONTINUE
      DO 250 I=1,NOCP
C
      READ(5,920) (GWW(I,J),J=1,NOCM)
C
```

```
C   GWW = DOWNWASH INFLUENCE COEFFICIENTS FROM PROGRAM I
  250 CONTINUE
C
      READ(5,910) LMAX,ALFA
C
C   LMAX = ORDER OF VORTEX APPROXIMATION
C   ALFA = ANGLE OF ATTACK (IN RADIANS)
       SINALF = SIN(ALFA)
      WRITE(6,960) LMAX,ALFA
C
C CALCULATE COLLOCATION POINTS
      CALL COLPT(NCORD,NSPAN,  XPT,YPT)
      IF(J1.EQ.1) GO TO 300
      XPT(NCORD2)=(XPT(NCORD)+XPT(NCORD-1))/2.
  300 CONTINUE
C
      READ (5,920) GYVOR,GZVOR
C
C   GYVOR ARE COEFFICNETS OF VORTEX SPANWISE LOCATION
C   GZVOR ARE COEFFICIENTS OF VORTEX VERTICAL LOCATION
      WRITE(6,890) GYVOR,GZVOR
C
C CALCULATES LOCATION OF VORTEX AT X =1.
      CALL FNCTN(GYVOR,LMAX,1.,YVOR)
      CALL FNCTN(GZVOR,LMAX,1.,ZVOR)
C
C CALCULATE CONTRIBUTION FROM LEADING-EDGE VORTEX TO DOWNWASH
      DO 400 I=1,NCORD2
      DO 400 J=1,NSPAN
      N1=J+(I-1)*NSPAN
      XPI=XLE(YPT(J))+B(YPT(J))*XPT(I)
      YPJ=YPT(J)*S
C
C OUTPUT LOCATION OF COLLOCATION POINTS
      WRITE(6,940) N1,XPT(I),YPT(J),XPI
      YDIFF=YVOR    -YPJ


      YSUM=YVOR    +YPJ
      XDIFF=1.-XPI
      YDIFSQ=YDIFF*YDIFF
      YSUMSQ=YSUM*YSUM
      ZSQ=ZVOR    *ZVOR
      TERM1=YDIFSQ+ZSQ
      TERM2=YSUMSQ+ZSQ
C
C CALCULATE CONTRIBUTION OF VORTEX AFT OF X =1.
      GWGMW2=-(YDIFF/TERM1*(1.-XDIFF/SQRT(TERM1+XDIFF*XDIFF))
     1+YSUM/TERM2*(1.-XDIFF/SQRT(TERM2+XDIFF*XDIFF)))/(4.*PI)
      DO 450 MQ=1,NCOM
      M=MQ-1
C
C OBTAIN CONTRIBUTION FROM WAKE AND VORTEX SEGMENT BEFORE X =1.
      CALL GAUS1D(   0.0,S,GWT,XGWT )
      CALL GAUS1D(   0.0,.13,GW,XGWGMW)
      GWGMW1=GW
      CALL GAUS1D(   .13,.25,GW,XGWGMW)
      GWGMW1=GWGMW1+GW
      CALL GAUS1D(   .25,.57,GW,XGWGMW)
      GWGMW1=GWGMW1+GW
      CALL GAUS1D(   .57,1.0,GW,XGWGMW)
      GWGMW1=GWGMW1+GW
      COEFF(N1,NMOD+MQ)=GWW(N1,MQ)+GWT*GWGMW1+GWGMW2
      GWGMW2=-1.*GWGMW2
  450 CONTINUE
  400 CONTINUE
      IF(J4.NE.1) GO TO 510
      NOCD=(NMODT+4)/5
C
C OUTPUT TOTAL DOWNWASH INFLUENCE COEFFICIENTS IF DESIRED
      DO 500 I=1,NOCP
      DO 500 K=1,NOCD
      JMAX=5*K
      JMIN=JMAX-4
```

```
      WRITE(6,980) (COEFF(I,J),J=JMIN,JMAX),I,K
      WRITE(7,980) (COEFF(I,J),J=JMIN,JMAX),I,K
  500 CONTINUE
  510 CONTINUE
C
C CALCULATES VORTICITY COEFFICIENTS A,GO FROM BOUNDARY CONDITION
C
C SQUARE MATRIX BY FORMING A TRANSPOSE*A
      CALL AXA(COEFF,PR,NOCP,NMODT)
      IF(J4.NE.1) GO TO 160
C
C OUTPUT A TRANSPOSE*A , IF DESIRED
      DO 150 L=1,NMODT
      WRITE(6,920) (PR(L,K),K=1,NMODT)
  150 CONTINUE
  160 CONTINUE
C
C FIX DOWNWASH, VR(I), ON WING
       DO 530 I=1,NOCP
      VR(I) = -SINALF
  530 CONTINUE
C
C FORM A TRANSPOSE*DOWNWASH VECTOR
      DO 140 K=1,NMODT
      ATVR(K)=0.0
      DO 140 L=1,NOCP
       ATVR(K)=ATVR(K)+COEFF(L,K)*VR(L)
  140 CONTINUE
      IF(J4.NE.1) GO TO 145
C
C OUTPUT A TRANSPOSE*DOWNWASH, IF DESIRED
      WRITE(6,920) (ATVR(I),I=1,NMODT)
  145 CONTINUE
      IF(NMODT.NE.NOCP) GO TO 130
C
C SOLVE SIMULTANEOUS LINEAR EQUATIONS, A X = B, FOR X


      CALL PRESS (COEFF,VR,NMODT,J4)
C
C SOLVE EQUATION ATA X = AT B FOR X
  130 CALL PRESS(PR,ATVR,NMODT,J4)
  860 FORMAT('  PROGRAM  III A  CALCULATES  A,GO:  GIVEN AWW,GWW:',5X,
     C'UPDATED APRIL 27,1977',/)
  870 FORMAT(' CHOW COLL PTS=',I3,3X,'SPNWS COLL PTS=',I3,3X,
     C'SEMISPAN=',F10.4,3X,'CHOWS MODES=',I3,3X,'SPNWS MODES=',I3,
     C 3X,'CR=',F7.4,/)
  880 FORMAT(2I10,F10.4,2I10,F10.4)
  890 FORMAT(' THE VALUES OF GYVOR,GZVOR ARE'/(5E14.5))
  900 FORMAT(5E14.6)
  910 FORMAT(I5,F10.6)
  920 FORMAT (5E14.5)
  940 FORMAT(' COLLOCATION POINT',I3,2F12.4,3X,'LOCAL X=', F12.4)
  950 FORMAT(4I5)
  960 FORMAT(1H0,I3,'  DEGREES OF FREEDOM IN VORTEX LOCATION',5X,
     C 'ANGLE OF ATTACK =',F7.4,/)
  980 FORMAT (5E14.5,2X,'COF',I2,1X,I2)
      STOP
      END
```

```
                FUNCTION XGWGMW(X)
C
C  XGWGMW CALCULATES DOWNWASH CONTRIBUTION FROM BOTH VORTICES
C
C       ARGUMENT LIST
C            X:   INTEGRATION POINT
C
      COMMON XPT,YPT,S,M,N
      PI=3.141593
      GGAM=SIN(FLOAT(2*M+1)/2.*PI*X)
      XGWGMW=GGAM*(FW(X,YPT)+FW(X,-YPT))
      RETURN
      END
```

```
                SUBROUTINE AXA(A,B,NROW,NCOL)
C
C  AXA CALCULATES B = A TRANSPOSE*A
C
C       ARGUMENT LIST
C            A:   INPUT MATRIX
C            B:   OUTPUT MATRIX
C            NROW:  NO. OF ROWS IN A TO BE PROCESSED
C            NCOL:  NO. OF COLUMNS IN A TO BE PROCESSED
C
C
      DIMENSION A(25,25),B(25,25)
      DO 10 I=1,NCOL
      DO 10 J=1,NCOL
      B(I,J)=0.0
      DO 10 N=1,NROW
      B(I,J)=A(N,I)*A(N,J)+B(I,J)
   10 CONTINUE
      RETURN
      END
```

```
                    SUBROUTINE DETERM(A,N)
C
C  DETERM CALCULATES DETERMINANT OF COFACTORS AS MATRIX INDICATOR
C
C         ARGUMENT LIST
C               A:    MATRIX TO BE TESTED
C               N:    ORDER OF MATRIX OF INTEREST
C
      DIMENSION A(25,25),DUMMY(25,25),IPER(25) ,DET(25,25)
      IER=0
C
C USE DUMMY TO PRESERVE ORIGINAL MATRIX AS SOLUTION PROCEDURE IS DESTRUCTIVE
      DO 100 I=1,N
      DO 100 J=1,N
  100 DUMMY(I,J)=A(I,J)
      WRITE(6,940) ((DUMMY(I,J),J=1,N),I=1,N)
  940 FORMAT( ' THE VALUES OF THE MATRIX ELEMENTS ARE'/(5E14.5))
      N1=N-1
C
C NOW DEVELOP PROCESS FOR COFACTORS
      DO 800 I=1,N
      DO 800 J=1,N
      DO 300 I1=1,N
       I2=I1
      IF(I1.GT.I) I2=I1-1
       DO 300 J1=1,N
      J2=J1
       IF(J1.GT.J) J2=J1-1
      DUMMY(I2,J2)=A(I1,J1)
  300 CONTINUE
      CALL MFG(DUMMY,25,N1,IPER,IS,IER)
C
C  MFG IS IBM SLMATH SUBROUTINE TO PERFORM LU DECOMPOSITION OF MATRIX
C
C         ARGUMENT LIST   MFG(A,M,N,IPER,IS,IER)
C


C               A:    INPUT MATRIX TO BE FACTORED
C               M:    ORDER OF MATRIX IN DIMENSION STATEMENT
C               N:    NUMBER OF SIMULTANEOUS EQUATIONS
C               IPER:  PERMUTATION VECTOR GENERATED FOR  MSG
C               IS:  SIGN OF DETERMINANT
C               IER: ERROR INDICATOR
C
      DET(I,J) = FLOAT(IS)
      DO 400 K=1,N1
      DET(I,J)= DUMMY(K,K)*DET(I,J)
  400 CONTINUE
  800 CONTINUE
      DO 600 I=1,N
  600 WRITE(6,930) I,(DET(I,J),J=1,N)
  930 FORMAT(' ROW', I5/(5E14.5))
      RETURN
      END
```

```
      SUBROUTINE PRESS(COEFF,SOLN,NMODT,J4)
C
C  PRESS CALCULATES VORTICITY COEFFICIENTS BY SOLUTION OF SIMULTANEOUS
C       EQUATIONS AND CALCULATES LEADING-EDGE VORTEX STRENGTH
C
C         ARGUMENT LIST
C              COEFF: A MATRIX IN  A X = B
C              SOLN:  B VECTOR IN  A X = B
C              NMODT: NUMBER OF SIMULTANEOUS EQUATIONS
C              J4:  CONTROLS PRINTING OF INTERMEDIATE RESULTS
C
      REAL*8  DOEFF(25,25),DSOLN(25)
      DIMENSION            XDUM(25),BPRIM(25),BDUM(25),COEFF(25,25),
     C SOLN(25),IPER(25),GAMMA(10)
      COMMON/MODES/NOCH,NOSH/CONTRL/J2,J3
      IER=0
      PI=3.141593
C
C INITIALIZE VORTICITY COEFFICIENTS XDUM(I)
       DO 10 I=1,NMODT
      XDUM(I)=0.0
      DO 10 J=1,NMODT
   10 DOEFF(I,J) = DBLE(COEFF(I,J))
      IF(J2.NE.1) GO TO 40
C
C DETERM CAN BE CALLED TO TEST FOR ILL-CONDITIONED MATRICES
       CALL DETERM(COEFF,NMODT)
   40 CONTINUE
      CALL DMFG(DOEFF,25,NMODT,IPER,IS,IER)
C
C DMFG IS IBM SLMATH SURROUTINE TO PERFORM LU DECOMPOSITION OF MATRIX
C
C         ARGUMENT LIST  DMFG(A,M,N,IPER,IS,IER)
C
C              A:   INPUT MATRIX TO BE FACTORED
C              M:   ORDER OF MATRIX IN DIMENSION STATEMENT
C              N:   NUMBER OF SIMULTANEOUS EQUATIONS
C              IPER:  PERMUTATION VECTOR GENERATED FOR DMSG
C              IS:  SIGN OF DETERMINANT
C              IER: ERROR INDICATOR
C
        DET = FLOAT(IS)
      DO 200 I=1,NMODT
      DET = DET*SNGL(DOEFF(I,I))
  200 CONTINUE
C
C OUTPUT DETERMINANT OF MATRIX
        WRITE(6,950) NMODT,DET
        IC=0
C
C  IC = CONTROL; COUNTS NUMBER OF ITERATIONS ALLOWED TO ELIMINATE
C        RESIDUE FROM AX = B SOLUTION
      IF(IER.EQ.0) GO TO 15
C
C  IER IS CONDITION PARAMETER PRODUCED BY DMFG. IER=0 IS BAD
      RETURN
   15 CONTINUE
      IC=IC+1
      DO 20 I=1,NMODT
      DSOLN(I) = SOLN(I)
   20 BDUM(I)=SOLN(I)
      CALL DMSG(DOEFF,25,IPER,NMODT,0,DSOLN,IER)
C
C DMSG IS IBM SLMATH SURROUTINE TO SOLVE SIMULTANEOUS LINEAR EQUATIONS
C       GIVEN LU DECOMPOSITION
C
C         ARGUMENT LIST  DMSG(A,M,IPER,N,J,B,IER)
C
C              A:   OUTPUT FROM DMFG
C              M:   ORDER OF MATRIX IN DIMENSION STATEMENT
C              IPER:  OUTPUT FROM DMFG
C              N:   NUMBER OF SIMULTANEOUS EQUATIONS
```

```
C              J:    NOMRALLY 0, TO OUTPUT X
C              B:    INPUT B; OUTPUTS X IN A X = B
C              IER: ERROR INDICATOR
      DO 100 I=1,NMODT
  100 SOLN(I) = SNGL(DSOLN(I))
      IF(IER.EQ.0) GO TO 70
C
C   IER   IS MATRIX CONDITION PARAMETER PRODUCED BY DMSG
      RETURN
   70 CONTINUE
      IF (J4.NE.1) GO TO 120
C
C OUTPUT INTERMEDIATE RESULTS IF DESIRED
      WRITE(6,960) (   SOLN(K),K=1,NMODT)
  120 CONTINUE
      NMOD=NOCM*NOSM
C
C CALCULATE VORTICITY COEFFICIENT VECTOR XDUM(I)
       DO 65 I=1,NMODT
   65 XDUM(I)=XDUM(I)+SOLN(I)
C
C CALCULATE LEADING-EDGE VORTEX STRENGTH, GAMMA(X)
      DO 90 II =1,10
      GAMMA(II) = 0.
       X = .1*FLOAT(II)
      DO 80 I=1,NOCM
      J=I+NMOD
      GAMMA(II) = XDUM(J)*SIN(FLOAT(2*I-1)/2.*PI*X) + GAMMA(II)
   80 CONTINUE
   90 CONTINUE
C
C OUTPUT LEADING-EDGE VORTEX STRENGTH
      WRITE(6,940) GAMMA
C
C OUTPUT VORTICITY COEFFICIENTS
      WRITE(6,920) (   XDUM(J),J=1,NMODT)



C
C CHECK NUMERICAL PROCEDURE BY CALCULATING B, FROM A*X
       DO 50 I=1,NMODT
       BPRIM(I)=0.0
      DO 50 J=1,NMODT
   50 BPRIM(I) = BPRIM(I) + COEFF(I,J) *SOLN(J)
C
C CHECK DIFFERENCE BETWEEN INITIAL B VECTOR AND CALCULATED B VECTOR
      DO 55 I=1,NMODT
   55 SOLN(I)=BPRIM(I)-BDUM(I)
      IF(J4.NE.1) GO TO 250
C
C OUTPUT CALCULATED B VECTOR, IF DESIRED
       WRITE(6,980) (BPRIM(I),I=1,NMODT)
  250 CONTINUE
C
C OUTPUT DELTA B
       WRITE(6,970) (SOLN(I),I=1,NMODT)
      IF(IC.LT.J3) GO TO 15
C
C PUNCH VORTICITY COEFFICIENTS
C
C        FIRST NOCM*NOSM MODES ARE HORSESHOE MODES; REMAINING MODES ARE
C        LEADING-EDGE VORTEX MODES
C
      WRITE(7,930) (XDUM(K),K=1,NMODT)
  920 FORMAT('OLOADING COEFFICIENTS ARE',/,(10E13.5))
  930 FORMAT(5E14.5,4X,'SOLN')
  940 FORMAT('OGAMMA AT X = .1,.2,.3,...,1.0',/,10F10.6)
  950 FORMAT(' DETERMINANT OF MATRIX OF ORDER',I5,2X,'IS',E12.5)
  960 FORMAT(' DEL X IS'/(5E14.5))
  970 FORMAT (' DEL B IS'/(5E14.5))
  980 FORMAT(' THE CALCULATED B VECTOR IS'/(5E14.5))
      RETURN
      END
```

C.4. Program III Prime

The following listing for Program III Prime includes Program III Prime and subprograms XGVGM, ADEL, and AGAM.

Program III Prime calculates the loading on the wing for a given vorticity distribution and vortex location.

```
C                    PROGRAM III PRIME
C
C  III PRIME CALCULATES LOADING ON WING: GIVEN VORTICITY
C
C   NEED FUNCTIONS P,FV,GVORT,XGVGM
C   NEED SUBROUTINES ADEL,AGAM,DFNCT,FNCTN,GAUS10
C
C        INPUT  LMAX,J4        2I5
C        INPUT  GYVOR,GZVOR        5E14.5
C        INPUT  NCORD,S,NOCM,NOSM,CR    I10,F10.4,2I10,F10.4
C        INPUT  XPT     5F10.6
C        INPUT  GVGAM        5E14.5    IF J4=1
C        INPUT  A,GO         5E14.5
C
      EXTERNAL XGVGM
       DIMENSION XPT(5),         YTEST(11),ADELT(5,5),AGAMM(5,5),
     C           A(5,5),GO(5),GVGAM(66,5)
      COMMON XPT,YPJ,S,M,MP/WING/CSR/VLOC/LMAX/PLAN/CR
      COMMON/GVOR/GYVOR(5),GZVOR(5)/SEC/ZVORT /GAUS/G(24),W(24)
C
      DATA YTEST/0.0,.3,.5,.6,.7,.75,.8,.85,.9,.95,1.0/,GVGAM/330*0./
      PI=3.141593
      ZVORT=0.0
      DO 150 JDUMMY =13,24
       W(JDUMMY) = W(25-JDUMMY)
  150 G(JDUMMY) = -G(25-JDUMMY)
C
      WRITE(6,890)
C
      READ(5,910) LMAX,J4
C
C  LMAX IS DEGREES OF FREEDOM IN VORTEX POSITION
C  J4 IS A CONTROL PARAMETER. IF(J4.EQ.1) INPUT GVGAM: ELSE CALCULATE
C
      READ(5,920) GYVOR,GZVOR
C


C  GYVOR: COEFFICIENTS FOR VORTEX SPANWISE POSITION
C  GZVOR: COEFFICIENTS FOR VORTEX VERTICAL POSITION
      WRITE(6,950) GYVOR,GZVOR
C
C CALCULATE VORTEX POSITION AT X=1
       CALL FNCTN(GYVOR,LMAX,1.,YVOR)
      CALL FNCTN(GZVOR,LMAX,1.,ZVOR)
C
      READ(5,880) NCORD,       S,NOCM,NOSM,CR
C
C  NCORD: NO. OF CHORDWISE POINTS
C  S: SEMISPAN; NON-D BY MAXIMUM LENGTH
C  NOCM: NO. OF CHORDWISE MODES
C  NOSM: NO. OF SPANWISE MODES
C  CR: ROOT CHORD; NON-D BY MAXIMUM LENGTH
      WRITE(6,870) NCORD,       S,NOCM,NOSM,CR
C
C CALCULATE CHORD TO SPAN RATIO, CSR
      CSR = CR/(2.*S)
C
C INPUT CHORDWISE POINTS OF INTEREST
C
      READ(5,960) XPT
       NC = 11*NCORD
      IF(J4.NE.1) GO TO 110
      DO 100 I=1,NC
C
      READ(5,920) (GVGAM(I,MQ),MQ=1,5)
  100 CONTINUE
  110 CONTINUE
C
      READ(5,920) ((A(I,J),I=1,NOCM),J=1,NOSM),(GO(K),K=1,NOCM)
C
C  A: HORSHOE VORTEX COEFFICIENTS
C  GO: LEADING-EDGE VORTEX COEFFICIENTS
      WRITE(6,990)((A(I,J),I=1,NOCM),J=1,NOSM),(GO(K),K=1,NOCM)
```

```
C
      WRITE(6,940)
C
C CALCULATE PRESSURES ALONG LINES X = XPT(I)
      DO 400 I=1,NCORD
      XPI=XPT(I)
      DO 400 J=1,11
      NI=J+(I-1)*11
      IF (XPI.GT.CR) GO TO 120
      YPJ=YTEST(J)*XPI*S
      GO TO 130
  120 CONTINUE
      YTE = S*(XPI-CR)/(1.-CR)
      YLE  = S*XPI
      YPJ = YTE+YTEST(J)*(YLE-YTE)
  130 ETA = YPJ/S
C
C ASSUMED ARROW WING FOR THETA
      THT = ARCOS(((2.-CR)*ETA -2.*XPI+CR)/(CR*(1.-ETA+.00001)))
C
C INITIALIZE SUMMATION VARIABLES
      GAMMA=0.0
      DELTA=0.0
      VGM=0.0
C
C CALCULATE LEADING-EDGE VORTEX CONTRIBUTIONS TO WING VORTICITY
      DO 350 MQ=1,NOCM
      M=MQ-1
      SGAMMA=GO(MQ)*XPI*GVORT(M,XPI,YPJ,S)
      GAMMA=SGAMMA+GAMMA
      SDELTA=-GO(MQ)*YPJ*GVORT(M,XPI,YPJ,S)
  350 DELTA = DELTA + SDELTA
      IF(J.EQ.11) GO TO 260
C
C CALCULATE HORSESHOE VORTEX CONTRIBUTIONS TO WING VORTICITY
      CALL ADEL (ADELT,THT,ETA,NOCM,NCSM)


      CALL  AGAM (AGAMM,THT,ETA,NOCM,NOSM)
      DO 300 MQ =1,NOCM
      DO 300 MPP=1,NCSM
      SGAMMA=A(MQ,MPP)*AGAMM(MQ,MPP)
      GAMMA=SGAMMA+GAMMA
      SDELTA=A(MQ,MPP)*ADELT(MQ,MPP)
  300 DELTA = DELTA + SDELTA
  260 CONTINUE
      IF(J4.EQ.1) GO TO 365
C
C CALCULATE CONTRIBUTION TO SPANWISE VELCOITY FROM LEADING-EDGE
C        VORTEX AFT OF X =1.
C        CALCULATE GVGM2
      XDIFF=1.-XPI
      YDIFF=YVOR     -YPJ
      YSUM=YVOR      +YPJ
      TERM1=YDIFF*YDIFF+ZVOR     *ZVOR
      TERM2=YSUM*YSUM+ZVOR     *ZVOR
      GVGM2=ZVOR      *(((1.-XDIFF/SQRT(TERM1+XDIFF*XDIFF))/TERM1-
     1((1.-XDIFF/SQRT(TERM2+XDIFF*XDIFF))/TERM2)/(4.*PI)
C
C CALCULATE  V
      DO 360 MQ=1,NOCM
      M=MQ-1
C
C CALCULATE CONTRIBUTION FROM VORTEX FORWARD OF X=1
      CALL GAUSID(   0.0,.13,GVGMS,XGVGM )
      GVGM1=GVGMS
      CALL GAUSID(   .13,.25,GVGMS,XGVGM )
      GVGM1=GVGMS+GVGM1
      CALL GAUSID(   .25,.57,GVGMS,XGVGM )
      GVGM1=GVGMS+GVGM1
      CALL GAUSID(   .57,1.0,GVGMS,XGVGM )
      GVGM1=GVGMS+GVGM1
      GVGAM(N1,MQ)=GVGM1+GVGM2
  360 GVGM2=-1.*GVGM2
```

```
  365 CONTINUE
      DO 370 MQ=1,NOCM
      VGM=GQ(MQ)*GVGAM(NI,MQ)+VGM
  370 CONTINUE
C
C CALCULATE PRESSURE DIFFERENCE COMPONENTS DUE TO SPANWISE AND
C      CHORDWISE COMPONENTS
      PV=2.*VGM*DELTA
      PU=-2.*GAMMA
      DELTP=PU+PV
C
C OUTPUT PRESSURE DIFFERENCE
      WRITE(6,930) XPI,YTEST(J),ETA,GAMMA,DELTA,PU,PV,DELTP
  400 CONTINUE
C
C OUTPUT VELOCITY CONTRIBUTIONS
      WRITE(6,900) ((GVGAM(I,J),J=1,NOCM),I=1,NC)
C
  870 FORMAT('OCHDWS PT' =',I3,3X,                       'SEMISPAN =',
     C F6.3,3X,'CHDWS MODES=',I3,3X,'SPNWS MODES=',I3,3X,'CR=',F7.4,/)
  880 FORMAT( I10,F10.4,2I10,F10.4)
  890 FORMAT(' LII PRIME;  UPDATED APRIL 28,1977',/)
  900 FORMAT('OTHE VALUES OF GVGAM ARE'/(10E13.4))
  910 FORMAT(2I5)
  920 FORMAT (5E14.5)
  930 FORMAT(3F7.4,5E14.5)
  940 FORMAT(1H0,T4,'XPI',T11,'YPJ',T18,'ETA',T28,'GAMMA',T42,'DELTA',
     C T57,'PU',T71,'PV',T84,'DELTP'/)
  950 FORMAT('0 THE VALUES OF GYVCR, GZVOR ARE'/(5E14.5))
  960 FORMAT (5F10.6)
  970 FORMAT('OCHDWS MODES =',I5,3X,'SPNWS MODES =',I5)
  990 FORMAT('OTHE VALUES OF A,GQ ARE'/(5E14.5))
      STOP
      END
```

```
                    FUNCTION XGVGM(X)
C
C  XGVGM CALCULATES CONTRIBUTION TO TG SPANWISE VELOCITY FROM VORTEX
C
C         ARGUMENT LIST
C
C               X:   CHORDWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N
      PI=3.141593
C
C CALCULATE LEADING-EDGE VORTEX STRENGTH
      GGAM=SIN(FLOAT(2*M+1)/2.*PI*X)
C
C CALCULATE SPANWISE VELOCITY CONTRIBUTION FROM LEADING-EDGE VORTICES
      XGVGM =GGAM    *(FV(X,YPT,XPT)-FV(X,-YPT,XPT))
      RETURN
      END
```

```
      SUBROUTINE ADEL (ADELT,THT,ETA,NOCM,NOSM)
C
C  GAUS9 CALCULATES ADELT: CONTRIBUTION TO CHRODWISE VORTICITY
C     FROM HORSESHOE VORTICES
C
C     ARGUMENT LIST
C
C           ADELT: CHORDWISE VORTICITY CONTRIBUTION
C           THT: ANGULAR CHORDWISE LOCATION
C           ETA: SPANWISE COORDINATE; NON-D BY SEMISPAN
C           NOCM:  NO. OF CHORDWISE MODES
C           NOSM:  NO. OF SPANWISE MODES
C
      COMMON /PLAN/CR
      DIMENSION CHDMOD(6),CHEBY2(10),ADELT(5,5)
C
      PI=3.141593
      CF = 2.*(2.-CR)/CR
      IF(ETA.GT..000001) GO TO 150
C
      DO 140 ICM=1,NOCM
      DO 140 JSM=1,NOSM
C
C FOR POINTS NEAR CENTERLINE, ZERO STRENGTH
      ADELT(ICM,JSM)=0.0
  140 CONTINUE
      GO TO 800
  150 CONTINUE
       THETA=THT
C
C USE CHEBYSHEV POLYNOMIALS FOR SPANWISE LOADING FUNCTIONS
C CALCULATE CHEBYSHEV POLYNOMIALS
      CHEBY2(1)=1.0
      CHEBY2(2)=2.*ETA
      NOSM2=2*NOSM-1
       IF(NOSM2.LT.3) GO TO 40
      CSQ=CHEBY2(2)
      DO 30 I=3,NOSM2
      CHEBY2(I)=CSQ*CHEBY2(I-1)- CHEBY2(I-2)
   30 CONTINUE
   40 CONTINUE
      IF(ETA.GE.1.) ETA=.999
C
C CALCULATE PRELIMINARY FACTORS
       FCTOR=-8.*PI     /SQRT(1.-ETA*ETA)
      NOCM2=NOCM+1
      CHDMOD(1)=THETA
      DO 500 ICM=1,NOCM2
      CHDMOD(ICM+1)=SIN(FLOAT(ICM)*THETA)/FLOAT(ICM)
  500 CONTINUE
C
      DO 600 ICM=1,NOCM
      ADELT(ICM,1)= FCTOR*(CHDMOD(ICM)-CHDMOD(ICM+2)
     C-FLOAT(ICM)*(1.+ETA)*(CF*CHDMOD(ICM+1)+CHDMOD(ICM)+CHDMOD(ICM+2)))
     C/FLOAT(2**(2*ICM))
      DO 600 JSM=2,NOSM
      ADELT(ICM,JSM)=FCTOR*(((1.-ETA* (2*(JSM-1)))*CHEBY2(2*JSM-1)
     C+FLOAT(2*JSM-1)*CHEBY2(2*JSM-2))*(CHDMOD(ICM)-CHDMOD(ICM+2))
     C-FLOAT(ICM)*(1.+ETA)*CHEBY2(2*JSM-1)*(CF*CHDMOD(ICM+1)+CHDMOD(ICM)
     C+CHDMOD(ICM+2)))/FLOAT(2**(2*ICM))
  600 CONTINUE
C
C        PRIMARY OUTPUT  ADELT PASSED TO CALLING PROGRAM THROUGH
C              ARGUMENT LIST
C
  800 RETURN
      END
```

```
                SUBROUTINE   AGAM (AGAMM,THT,ETA,NOCM,NOSM)
C
C  GAUSIO CALCULATES SPANWISE VORTICITY CONTRIBUTION FROM
C      HORSESHOE VORTICES
C
C        ARGUMENT LIST
C
C             AGAMM: SPANWISE VORTICTY CONTRIBUTION
C             THT: ANGULAR CHORDWISE LOCATION
C             ETA: SPANWISE COORDINATE; NON-D BY SEMISPAN
C             NOCM:  NO. OF CHORDWISE MODES
C             NOSM:  NO. OF SPANWISE MODES
C
      COMMON/WING/CSR
      DIMENSION AGAMM(5,5),CHEBY2(5),CHOMOD(5)
C
      PI=3.141593
C
C CALCULATE CHEBYSHEV POLYNOMIALS
       CHEBY2(1)=1.0
      CHEBY2(2)=4.*ETA*ETA-1.
      IF(NOSM.LT.3) GO TO 40
        CSQ=CHEBY2(2)-1.
      DO 30 J=3,NOSM
      CHEBY2(J)=CSQ*CHEBY2(J-1)-CHEBY2(J-2)
   30 CONTINUE
   40 CONTINUE
C
C PREVENT BLOWING UP
       IF(ETA.GE.1.) ETA = .999
      THETA=THT
C
C CALCULATE INTERMEDIATE FACTORS
      FACTOR = 16.*PI/(CSR*B(1,ETA))*SQRT(1.-ETA*ETA)
C
      DO 70 ICM=1,NOCM
      CHOMOD(ICM)=SIN(FLOAT(ICM)*THETA)/FLOAT(2**(2*ICM))
   70 CONTINUE
      DO 60 ICM=1,NOCM
      DO 60 JSM=1,NOSM
      AGAMM(ICM,JSM)=FACTOR*CHOMOD(ICM)*CHEBY2(JSM)
   60 CONTINUE
C
C        PRIMARY OUTPUT  AGAMM PASSED THROUGH ARGUMENT LIST
C             TO CALLING PROGRAM
C
      RETURN
      END
```

C.5. Program V

The following listing of Program V includes Program V and subprograms BLOCK DATA, A1, A5, B, XLE, B5, B7, DIDY, DIDZ, FV, FW, GVORT, XGVL, XGVT, XGWL, XGWT, CHDWS, COLPT, DFNCT, DGWGM, DGWV, FNCTN, FUNCTN, GAUS1D, GVCTR, GWVD, KERNL, TUCHEB, VORINT, WOV, and WPDW.

Program V calculates the new vortex position and vorticity distribution for a given initial solution given the outputs of Program I, Program WOW, and Program IIIA.

```
C                     PROGRAM V
C
C   PROGRAM V  CALCULATES NEW VORTEX POSITION AND VORTICITY DISTRI-
C        BUTION, USING DOWNWASH AND FORCE CONDITIONS
C
C          INPUT  J3,J4          2I10
C          INPUT  NCORD,NSPAN,S,NOCM,NOSM,CR        2I10,F10.4,2I10,F10.4
C          INPUT  NCFP,LMAX,FACTOR,ALFA,S2          2I10  3F10.6
C          INPUT  GYVOR,GZVOR                  5E14.5
C          INPUT  A,GO           5E14.5
C          INPUT  AWW,GWW        5E14.5
C          INPUT  NI(1),NI(2),NI(3),NI(4),NCP,J1,J2,ETA       5I5,2I2,F10.4
C
C    NEED FUNCTIONS: A1,A5,B,B5,B7,DIDY,DIDZ,FV,FW,GVORT,XGVL,XGVT,
C         XGWL,XGWT,XI,XIF
C    NEED SUBROUTINES: CHDWS,COLPT,DFNCT,DGWGM,DGWV,       FNCTN,FUNCTN,
C         GAUSLD,GVCIR,GWVD,KERNL,TUCHEB,VORINT,WOV,WPOW    BLOCK DATA
C
      DIMENSION XVECT(5,5),ATA(35,35),PARAM(35),FMINS(35),IPER(35)
      COMMON XPI,YVORT,S,M,MP /PLAN/CXMAX /WOW1/JS,NI(4)
     C        /CWPOW/ NCORD,NSPAN,COEFF(25,25),GWW(25,5),VR(25)
     C /WOV2/AR(4,5,5),ALS(5,5),NINC,CR(5),TKR(10),XO.,SOS,Y,Z,YMN,ZMZ,
     C RSOR,ETA,GAUSX(10),PO2,NCP,MP,N,X,GZ,J1,J2,CS,YMN2,ZMZ2,CSR
     C        /SEC/ZVORT/VLOC/LMAX/MODES/NOCM,NOSM /CONTR2/J3,J4
     C       /VORT/YVOR,/VOR/GVOR/GYVOR(5),GZVOR(5)
     C /GAUS/G(24),W(24)/YACOB/XACOB(35,35),SAWW(5,5),SAVW(5,5),
     C DAWDY(5,5),DAWDZ(5,5),DAVDY(5,5),DAVDZ(5,5)
     C /GVEC/ A(5,5),GO(5),N1,FSUBY(5),FSUBZ(5),PI,SINALF,NOFP
      REAL*8              CARAM(35),DACOB(35,35)
C
C INITIALIZE GAUSSIAN QUADRATURE WEIGHTS AND ABSCISSAS
       DO 150 JDUMMY = 13,24
      W(JDUMMY)=W(25 -JDUMMY)
  150 G(JDUMMY)=-G(25 -JDUMMY)
C
C INITIALIZE NO-FORCE POINTS


      DATA XVECT/.6,4*0.,.32,.84,3*0.,.22,.6,.9,12*0./
      PI = 3.141593
       WRITE(6,830)
C
      READ(5,880) J3,J4
C
C   J3: CONTROL PARAMETER. IF J3=1,NCORD2=NCORD; ELSE, NCORD2=NCORD+1
C   J4 CONTROLS OUTPUT; IF J4.EQ.1, OUTPUTS INTERMEDIATE RESULTS
      WRITE(6,880) J3,J4
C
      READ(5,880)  NCORD,NSPAN,S,NOCM,NOSM,CXMAX
C
C   NCORD = NO. OF CHORDWISE COLLOCATION POINTS
C   NSPAN = NO. OF SPANWISE COLLOCATION POINTS
C   S = SEMISPAN; NON-D BY MAXIMUM LENGTH
C   NOCM =NO. OF CHORDWISE LOADING MODES
C   NOSM =NO. OF SPANWISE LOADING MODES
C   CXMAX = ROOT CHORD; NON-D BY MAXIMUM LENGTH
      WRITE(6,870) NCORD,NSPAN,S,NOCM,NOSM,CXMAX
C
      READ (5,890) NOFP,LMAX,FACTOR,ALFA,S2
C
C   NOFP=NO. OF FORCE POINTS
C   LMAX = ORDER OF VORTEX APPROXIMATION
C   FACTOR: LIMITS CHANGES IN VORTEX POSITION
C   ALFA = ANGLE OF ATTACK (IN RADIANS)
C   S2: LIMITS CHANGES IN VORTICITY COEFFICIENTS
       WRITE(6,900) LMAX,FACTOR,ALFA,NOFP
C
C CALCULATE DOWNWASH
      SINALF=SIN(ALFA)
C
      READ(5,940) GYVOR,GZVOR
C
C   GYVOR= COEFFICIENTS OF HORIZONTAL VORTEX LOCATION
C   GZVOR= COEFFICIENTS OF VERTICAL VORTEX LOCATION
```

```
C
      WRITE(6,950) GYVOR,GZVOR
C
      READ (5,940)  ((A(I,J),I=1,NOCM),J=1,NOSM),(GO(K),K=1,NOCM)
C
C  A: HORSESHOE VORTICITY COEFFICIENTS
C  GO: LEADING-EDGE VORTICITY COEFFICIENTS
      WRITE(6,920)  ((A(I,J),I=1,NOCM),J=1,NOSM),(GO(K),K=1,NOCM)
C
      NMOD=NOSM*NOCM
      NMODT=NMOD+NOCM
       NOCP = NMCOT
      NOPTS = 2*NOEP + NOCP
      NMOD2= NMODT+ 2*LMAX
C
C  NMOD = NO. OF HORSESHOE VORTEX MODES
C  NMODT= TOTAL NO. OF VORTICITY MODES
C  NOCP = NO. OF COLLOCATION POINTS ON WING
C  NOPTS = TOTAL NO. OF CONTROL POINTS
C  NMOD2 = TOTAL NO. OF MODES
      DO 200 I=1,NOCP
C
      READ(5,940)  (COEFF(I,J),J=1,NMOD)
C
C  (COEFF(I,J),J=1,NMOD):  OUTPUT FROM PROGRAM WOW
       DO 200 J=1,NMOD
  200 COEFF(I,J) = -COEFF(I,J)
      DO 250 I=1,NOCP
C
  250 READ(5,940)  (GWW(I,J),J=1,NOCM)
C
C  GWW:  OUTPUT FROM PROGRAM I
C
      NINC=-1
C
C  NINC IS A CONTROL PARAMETER TO LIMIT REPETITIONS IN CHOWS


       JS=4
C
C  JS =NO. OF INTEGRATION REGIONS IN SPANWISE DIRECTION
C
    5  READ 501, NI(1),NI(2),NI(3),NI(4),NCP,J1,J2,ETA
C
C   NI(J)=NO. OF LEGENDRE-GAUSS POINTS IN SPANWISE INTEGRATION
C    IN REGION J
C   NCP=NO. OF CHORDWISE LEGENDRE-GAUSS QUADRATURE POINTS IN CHOWS
C   J1,J2 CONTROL OUTPUT; NORMALLY 0
C   ETA=ZETA=SMALL INTEGRATION REGION ABOUT SINGULARITY
      WRITE(6,970) NI(3),NCP,NI(1)
      SPAN = 2.*S
      CSR = CXMAX/SPAN
C
C  SPAN = WING SPAN; NON-D BY MAXIMUM LENGTH
C  CSR =CHORD TO SPAN RATIO
      N=1
C
C   N=SECTION NO. INDICATOR
C
C FORM SCALES TO NORMALIZE EQUATIONS
      F1 = 2.*ALFA/(PI*PI)
      F2 = ALFA*PI*S
      F3 = S
      F4 = ALFA/4.
C
C  ITRMAX = MAXIMUM NO. OF ITERATIONS ALLOWED TO CONVERGE
      ITRMAX = 15
C
C LOOP TO SATISFY DOWNWASH AND NO-FORCE CONDITIONS
      DO 800 ITR=1,ITRMAX
C
C CALCULATE VORTEX LOCATION AT X = 1.
      CALL FNCTN(GYVOR,LMAX,1.,YVOR)
      CALL FNCTN(GZVOR,LMAX,1.,ZVOR)
```

```
C
C FORMULATE DOWNWASH CONDITIONS ON WING
       CALL WDOWN(A,GQ,NOCP,ALFA)
C
C ADD CONTRIBUTIONS FROM DOWNWASH CONDITION TO RESIDUE VECTOR
      DO 60 I=1,NOCP
   60 PARAM(I) = -VR(I)
      NI= NOCP
C
C FORMULATE NO-FORCE CONDITION ON RIGHT-HAND VORTEX
      DO 400 L=1,NOFP
C
C FIND CONTROL POINT ON VORTEX
      XPI = XVECT(L,NOFP)
       CALL FNCTN(GYVOR,LMAX,XPI,YVORT)
      CALL FNCTN(GZVOR,LMAX,XPI,ZVORT)
C
C  XPI = CHORDWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C  YVORT: SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C  ZVORT = VERTICAL COORDINATE; NON-D BY MAXIMUM LENGTH
      XOC = XPI/CXMAX
      SOS = YVORT/S
      Z = ZVORT/S
C
C  XOC = CHORDWISE POSITION; NON-D BY ROOT CHORD
C  SOS = SPANWISE POSITION; NON-D BY SEMISPAN
C  Z = VERTICAL POSITION; NON-D BY SEMISPAN
C
C CALCULATE CONTRIBUTIONS TO FORCES AND RELATED DERIVATIVES FROM
C        HORSESHOE VORTICES
      CALL WOVIL)
C
C CALCULATE FORCES AND REMAINING DERIVATIVES FOR JACOBIAN
      CALL GVCTR(NOCP)
  400 CONTINUE
C


C ADD CONTRIBUTIONS FROM FORCE CONDITION TO RESIDUE VECTOR
      DO 500 J=1,NOFP
      PARAM(J+NOCP) = -      FSUBY(J)
      PARAM(J+NOFP+NOCP) = -FSUBZ(J)
  500 CONTINUE
C
C CALCULATE MAGNITUDE OF RESIDUE
      DMAG =0.
C
C SCALES DEPENDENT VARIABLES TO ORDER 1
      DO 560 IA = 1,NUPTS
      DMAG = DMAG + PARAM(IA)*PARAM(IA)
      DO 420 N2 = 1,NMOD
  420 XACOB(IA,N2) = XACOB(IA,N2) *F1
      DO 430 N2 = 1,NOCP
  430 XACOB(IA,N2+NMOD) = XACOB(IA,N2+NMOD)*F2
      DO 450 N2 =1,LMAX
  440 XACOB(IA,N2+NMODT) = XACOB(IA,N2+NMODT) *F3
  450 XACOB(IA,N2+LMAX+NMODT) = XACOB(IA,N2+LMAX+NMODT)*F4
  560 CONTINUE
C
C FORM MATRIX A TRANSPOSE*A
C       OBTAIN A TRANSPOSE * A MATRIX FOR STABILITY REASONS
C
C SAVE DACOB IN ATA SINCE SOLUTION PROCEDURE IS DESTRUCTIVE
      DO 360 I=1,NMOD2
      DO 360 J=1,NMOD2
       DACOB(I,J) = 0.D0
      DO 340 K= 1,NOPTS
      DACOB(I,J) = DACOB(I,J) + DBLE(XACOB(K,I)*XACOB(K,J))
  340 CONTINUE
  360 ATA(I,J) = SNGL(DACOB(I,J))
C
C PERFORM LU DECOMPOSITION OF MATRIX DACOB
  530 CALL DMFG(DACOB,35,NMOD2,IPER,IS,IFX)
C
```

```
C   DMFG IS IBM SLMATH SUBROUTINE
C
C CALCULATE DETERMINANT OF ATA MATRIX
      DET = FLOAT(IS)
C
C CALCULATE AT*B VECTOR
      DO 320 I=1,NMOD2
       DET = DET*SNGL(DACOB(I,I))
      DARAM(I) = 0.D0
      DO 320 J=1,NOPTS
  320 DARAM(I) = DARAM(I) + DBLE(XACOB(J,I)*PARAM(J))
C
C OUTPUT DETERMINANT OF ATA MATRIX
      WRITE(6,860) DET
C
C AS CHECK FOR SOLUTION PROCEDURE, COMPARE DARAM WITH FMINS
C
C OUTPUT A TRANSPOSE * B VECTOR
      WRITE(6,940) (DARAM(IL),IL =1,NMOD2)
C
C SOLVE SIMULTANEOUS LINEAR EQUATIONS
      CALL DMSG(DACOB,35,IPER,NMOD2,0,DARAM,IER)
C
C   SUBROUTINE DMSG IS IBM SLMATH SUBROUTINE
C
C CALCULATE A TRANSPOSE A * X VECTOR
      DO 570 IB =1,NMOD2
      FMINS(IB) = 0.
       DO 570 JB =1,NMOD2
  570 FMINS(IB) = FMINS(IB) + ATA  (IB,JB)*SNGL(DARAM(JB))
C
C OUTPUT CALCULATED A TRANSPOSE A * X VECTOR
      WRITE(6,940) (FMINS(ID),ID=1,NMOD2)
C
C OUTPUT PREDICTED CHANGES IN X IN AT A X = AT B
       WRITE(6,940) (DARAM(IE),IE=1,NMOD2)


C
C CALCULATE RELATIVE MAGNITUDE OF PREDICTED CHANGE IN VORTEX POSITION
      DEL1=FACTOR/SNGL(DSQRT(DARAM(1+NMODT)**2+DARAM(1+LMAX+NMODT)**2))
      IF(DEL1.GT.1. )  DEL1=1.
C
C CALCULATE RELATIVE MAGNITUDE OF PREDICTED CHANGES IN HORSESHOE
C      VORTICITY COEFFICIENTS
      ANUM=0.
      DNUM=0.
      DO 620 IK=1,NOCM
      DO 620 JK=1,NOSM
      ANUM= ANUM+A(IK,JK)*A(IK,JK)
      DNUM = DNUM+ SNGL(DARAM(IK+(JK-1)*NOCM))**2
  620 CONTINUE
      DEL2 = S2*SQRT(ANUM/DNUM)/F1
C
C USE SMALLER OF THE TWO SCALES
      IF (DEL1.GT.DEL2) DEL1= DEL2
C
C OUTPUT SCALES
      WRITE(6,930) S2,DEL1
C
C CALCULATE NEW VORTICITY COEFFICIENTS
      DO 600 I=1,NOCM
      GO(I) = GO(I) + DEL1*SNGL(DARAM(NMOD+I))*F2
      DO 600 J=1,NOSM
      A(I,J) = A(I,J) + DEL1*SNGL(DARAM(I+(J-1)*NOCM)) *F1
  600 CONTINUE
C
C CALCULATE NEW VORTEX LOCATION COEFFICIENTS
      DO 550 I=1,LMAX
       GYVOR(I) = GYVOR(I)+DEL1*SNGL(DARAM(I+NMODT))*F3
  550  GZVOR(I) = GZVOR(I)+DEL1*SNGL(DARAM(I+LMAX+NMODT))*F4
C
C OUTPUT NEW VORTICITY COEFFICIENTS
      WRITE(6,920)  ((A(I,J),I=1,NOCM),J=1,NOSM),(GO(K),K=1,NOCM)
```

```
C
C OUTPUT NEW VORTEX LOCATION COEFFICIENTS
      WRITE(6,950) GYVCR,GZVOR
      DMAG = SQRT(DMAG)
C
C OUTPUT ITERATION NO. AND RESIDUE
      WRITE(6,980) ITR,DMAG
C
C CHECK FOR CONVERGENCE
       IF (DMAG.LT..001) GO TO 810
  800 CONTINUE
  810 CONTINUE
C
C OUTPUT JACOBIAN
C
       WRITE(6,910) ((XACOB(IG,JG),JG=1,NMOD2),IG=1,NOPTS)
C
C PUNCH NEW VORTEX LOCATION COEFFICIENTS
      WRITE(7,840) GYVCR
      WRITE(7,850) GZVCR
C
C PUNCH NEW VORTICITY COEFFICIENTS
      WRITE(7,940)  ((A(I,J),I=1,NOCM),J=1,NOSM),(GQ(K),K=1,NOCM)
C
  501 FORMAT(5I5,2I2,F10.4)
  830 FORMAT(' PROGRAM V CALCULATES VORTICITY COEFFICIENTS AND VORTEX LO
     CCATION FROM INITIAL GUESS    APRIL 29,1977',//)
  840 FORMAT(5E14.5,1X,'GY( 1- 5)')
  850 FORMAT(5E14.5,1X,'GZ( 1- 5)')
  860 FORMAT(' DETERMINANT OF THE JACOBIAN IS=',E14.5)
  870 FORMAT('0CHOWS PTS =',I3,3X,'SPNWS PTS =',I3,3X,'SEMISPAN =',
     CF6.3,3X,'CHOWS MODES =',I3,3X,'SPNWS MODES =',I3,3X,'CR=',F7.4)
  880 FORMAT(2I10,F10.4,2I10,F10.4)
  890 FORMAT(2I5,3F10.6)
  900 FORMAT ('0 ORDER OF VORTEX APPROXIMATION IS',I3,3X,'FACTOR=',
     CF10.6,3X,'ANGLE OF ATTACK=',F10.6,3X,'NOFP=',I3)


  910 FORMAT(10 E13.4)
  920 FORMAT('0THE VALUES OF A,GQ ARE'/,(5E14.5))
  930 FORMAT(' SCALE FROM CHANGE IN LOADING COEFFICIENTS=',F10.4,5X,
     C 'DELT=',F10.5,/)
  940 FORMAT(5E14.5)
  950 FORMAT('0THE VALUES OF GYVOR,GZVOR ARE'/ (5E14.5))
  970 FORMAT(1H ,'INTEGRATION PTS. IN REGION 3=',I5,5X,'IN CHOWS=',I5,
     C5X,'IN REGION 1=',I5)
  980 FORMA ('0 AFTER',I3,' ITERATIONS, RESIDUE IS =',F10.6,/)
      STOP
      END
```

```
                   BLOCK DATA
C
C    GAUSSIAN  QUADRATURE ABSCISSA AND WEIGHTS
C
      COMMON/GAUN/GN(10,10),WN(10,10) /GAUS/ G(24),W(24)
C
      DATA GN/40*0.,
     1-.9061798,-.5384693,0.0      , .5384693, .9061798,45*0.,
     2-.9739065,-.8650634,-.6794096,-.4333954,-.1488743,
     3 .1488743, .4333954, .6794096, .8650634, .9739065/,WN/40*0.,
     4 .2369269, .4786287, .5688889, .4786287, .2369269,45*0.,
     5 .0666713, .1494513, .2190864, .2692667, .2955242,
     6 .2955242, .2692667, .2190864, .1494513, .0666713/
      DATA G/-.9951872,-.9747286,-.9382746,-.8864155,-.8200020,-.7401242
     1,-.6480936,-.5454215,-.4337935,-.3150427,-.1911189,-.0640569,12*0.
     2/,W/.0123412,.0285314,.0442774,.0592986,.0733465,.0861902,
     3.0976186,.1074443,.1155057,.1216705,.1258374,.1279382,12*0.0/
      END
```

```
                 FUNCTION AI(Y)
C
C  AI(Y) PROVIDES LOWER LIMIT FOR SURFACE INTEGRAL
C          ARROW WING CONFIGURATION
C
C      ARGUMENT LIST
C          Y:   SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N
      AI = ABS(Y)/S
      RETURN
      END
C
C*********************************************************************
C
                 FUNCTION A5(Y)
C
C  A5(Y) PROVIDES LOWER LIMIT FOR SURFACE INTEGRAL
C          ARROW WING CONFIGURATION
C
C      ARGUMENT LIST
C          Y:   SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N
      Y1 = ABS(Y)/S
       IF (Y1.GT.(XPT+.02)) GO TO 20
   10 A5 = XPT+.02
      RETURN
   20 A5 = Y1
      RETURN
      END
```

```
               FUNCTION B(N,S)
C
C   B: SEMICHORD NONDIMENSIONALIZED BY ROOT SEMICHORD
C               ARROW WING CONFIGURATION
C
C          ARGUMENT LIST
C               N:    SECTION NO. INDICATOR
C               S:    SPANWISE COORDINATE; NON-D BY SEMISPAN
C
      B=1.-ABS(S)
       RETURN
       END
C
C**************************************************************************
C
               FUNCTION XLE(N,S)
C
C  XLE: DESCRIBES LEADING EDGE OF WING; NON-D BY WING ROOT SEMICHORD
C               ARROW WING CONFIGURATION
C
C               ARGUMENT LIST
C               N:    SECTION NO. INDICATOR
C               S:    SPANWISE COORDINATE; NON-D BY SEMISPAN
C
      COMMON/PLAN/CR
      XLE = -1.+2.*ABS(S)/CR
      RETURN
      END
```

```
               FUNCTION B5(Y)
C
C  B5 PROVIDES PLANFORM LIMITS TO INTEGRATION ROUTINE
C               ARROW WING CONFIGURATION
C
C          ARGUMENT LIST
C               Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N /PLAN/CR
      B5 = ABS(Y)*(1.-CR)/S +CR
       RETURN
       END
C
C**************************************************************************
C
               FUNCTION B7(Y)
C
C B7 PROVIDES PLANFORM LIMITS FOR INTEGRATION ROUTINE
C               ARROW WING CONFIGURATION
C
C          ARGUMENT LIST
C               Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,N /PLAN/CR
       IF (Y.GT.S*(XPT+.02-CR)/(1.-CR)) GO TO 20
        B7 = ABS(Y)*(1.-CR)/S+CR
       RETURN
   20 B7 = XPT+.02
      RETURN
      END
```

```
                 FUNCTION DIDY(Y,YVORT)
C
C DIDY   PROVIDES DERIVATIVE FOR JACOBIAN
C
C         ARGUMENT LIST
C              Y:    SPANWISE COORDINATE: NON-D BY MAXIMUM LENGTH
C              YVORT: VORTEX SPANWISE POSITION: NON-D BY MAX. LENGTH
C
C FACTOR OF  T(2*L-1)  OUTSIDE OF FUNCTION
      COMMON XPI,YDUM,S,M,MP /SEC/ ZVORT /PLAN/CR
      YDIFF=Y-YVORT
       XEDGE = CR+Y*(1.-CR)/S
       XDIFF = XEDGE-XPI
      TERM1=YDIFF*YDIFF+ZVORT*ZVORT
      TERM3=TERM1*XDIFF*XDIFF
      DIDY =            YDIFF/TERM1*(2./TERM1*(1.-XDIFF/SQRT(TERM3))
     C -XDIFF/(TERM3*SQRT(TERM3)))
      RETURN
      END
```

```
                 FUNCTION DIDZ(Y,YVORT)
C
C DIDZ PROVIDES DERIVATIVE FOR JACOBIAN
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              Y:    SPANWISE COORDINATE: NON-D BY MAXIMUM LENGTH
C              YVORT: VORTEX SPANWISE POSITION: NON-D BY MAX. LENGTH
C
C FACTOR OF  T(2*L-1)  OUTSIDE OF FUNCTION
C
C USE MULTIPLE DEFINITION TO REDUCE TRUNCATION ERRORS
      REAL*8 YDIFF,XDIFF,TERM1,TERM2,A
      COMMON XPI,YPT,S,M,MP /PLAN/CR /SEC/ZVORT
       YDIFF = DBLE(Y-YVORT)
       XEDGE = CR+Y*(1.-CR)/S
      XDIFF = DBLE(XEDGE-XPI)
      TERM1 = YDIFF*YDIFF+DBLE(ZVORT*ZVORT)
      A=TERM1/(XDIFF*XDIFF)
      IF(A.LE..005D0) GO TO 100
      TERM2=TERM1+XDIFF*XDIFF
      DIDZ = ZVORT*SNGL((-2.D0/TERM1*(1.D0-XDIFF/DSQRT(TERM2))+
     C XDIFF/(TERM2*DSQRT(TERM2)))/TERM1)
      RETURN
  100 DIDZ = ZVORT/4.*SNGL((-3.D0+5.D0*A)/XDIFF**4)
      RETURN
      END
```

```
                  FUNCTION FV(X,YPT,XPT)
C
C   FV GIVES CONTRIBUTION FROM LEADING-EDGE VORTEX TO V
C
C         ARGUMENT LIST
C               X:    CHORDWISE INTEGRATION POINT; NON-D BY MAXIMUM LENGTH
C               YPT:  SPANWISE LOCATION OF CONTROL POINT
C               XPT:  CHORDWISE CONTROL POINT; NON-D BY MAXIMUM LENGTH
C LET XPT =0. TO USE ON WING
      COMMON/GVOR/ GYVOR(5),GZVOR(5) /SEC/ZPT /VLOC/LMAX
      PI=3.141593
C
C CALCULATE LOCATION OF VORTEX
      CALL FNCTN(GYVOR,LMAX,X,YVORT)
      CALL FNCTN(GZVOR,LMAX,X,ZVORT)
      CALL DFNCT (GZVOR,LMAX,X,DZVORT)
      XDIFF=X-XPT
      YDIFF=YVORT-YPT
      ZDIFF=ZVORT-ZPT
      FV    =(ZDIFF-XDIFF*DZVORT)/(4.*PI* (XDIFF*XDIFF+YDIFF*YDIFF+
     1ZDIFF*ZDIFF)**1.5)
      RETURN
      END
```

```
                  FUNCTION FW(X,YPT)
C
C   FW GIVES CONTRIBUTION OF LEADING-EDGE VORTEX TO W
C
C         ARGUMENT LIST
C               X:    CHORDWISE INTEGRATION POINT; NON-D BY MAXIMUM LENGTH
C               YPT:  SPANWISE CONTROL POINT; NON-D BY MAXIMUM LENGTH
C
C TO USE ON WING, LET ZPT=0.0
      COMMON XPT,YDUM,S,M,N /SEC/ZPT /VLOC/LMAX
      COMMON/GVOR /GYVOR(5),GZVOR(5)
      PI=3.141593
C
C CALCULATE LOCATION OF LEADING-EDGE VORTEX
      CALL FNCTN(GYVOR,LMAX,X,YVORT)
      CALL FNCTN(GZVOR,LMAX,X,ZVORT)
      CALL DFNCT(GYVOR,LMAX,X,DYVORT)
      XDIFF=X-XPT
      YDIFF=YVORT-YPT
      ZDIFF=ZVORT-ZPT
      FW  = (XDIFF*DYVORT-YDIFF)/((XDIFF*XDIFF+YDIFF*YDIFF+ZDIFF*ZDIFF)
     1**1.5* 4.*PI)
      RETURN
      END
```

```
                FUNCTION GVORT(M,X,Y,S)
C
C  GVORT CALCULATES VORTICITY STRENGTH ON WING DUE TO LEADING-EDGE
C       VORTICES
C
C        ARGUMENT LIST
C             M:   MODAL SPECIFICATION PARAMETER
C             X:   CHORDWISE POINT OF INTEREST; NON-D BY MAX. LENGTH
C             Y:   SPANWISE POINT OF INTEREST; NON-D BY MAX. LENGTH
C             S:   SEMISPAN; NON-D BY MAXIMUM LENGTH
C
      PI=3.141593
      CONST = PI*FLOAT(2*M+1)/2.
      X2Y2=SQRT(X*X+Y*Y)
      XEDGE=X2Y2/SQRT(1.+S*S)
      GVORT=CONST*COS(CONST*XEDGE)/X2Y2
      RETURN
      END
```

```
                FUNCTION XGVL(X)
C
C  XGVL CALCULATES CONTRIBUTION TO V FROM LEFT-HAND VORTEX
C
C        ARGUMENT LIST
C             X:   CHORDWISE INTEGRATION POINT; NON-D BY MAX. LENGTH
C
      COMMON XPT,YPT,S,M,MP
      PI=3.141593
      CONST=FLOAT(2*M+1)/2.*PI
      XGVL  =-SIN(CONST*X)*FV(X,-YPT,XPT)
      RETURN
       END
C
C*************************************************************************
C
                FUNCTION XGVT(Y)
C
C  XGVT GIVES CONTRIBUTION OF WAKE VORTICITY TO SPANWISE VELOCITY
C             ARROW WING CONFIGURATION
C
C        ARGUMENT LIST
C             Y:   SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
      COMMON XPT,YPT,S,M,MP/SEC/ZPT /PLAN/CR
      PI=3.141593
      CONST=PI*FLOAT(2*M+1)/2.
       XEDGE = CR*Y*(1.-CR)/S
      X2Y2 = SQRT(XEDGE**2+Y*Y)
      XEDGE=X2Y2/SQRT(1.+S*S)
      GDELT=-CONST*COS(CONST*XEDGE)
      XGVT  =-ZPT*GDELT*(XI(Y,YPT)-XI(Y,-YPT))/(4.*PI)
      RETURN
      END
```

```
                  FUNCTION XGWL(X)
C
C  XGWL CALCULATES CONTRIBUTION TO W FROM LEFT-HAND VORTEX
C
C         ARGUMENT LIST
C              X:    CHORDWISE INTEGRATION POINT; NON-D BY MAX. LENGTH
C
      COMMON XPT,YPT,S,M,N
      PI=3.141593
      CONST=FLOAT(2*M+1)/2.*PI
      XGWL  =SIN(CONST*X)*FW(X,-YPT)
      RETURN
      END
C
C**********************************************************************
C
                  FUNCTION XGWT(Y)
C
C XGWT GIVES CONTRIBUTION OF WAKE VORTICITY TO DOWNWASH
C
C         ARGUMENT LIST
C              Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C
C TO USE ON WING, LET ZPT=0.0
      COMMON XPT,YPT,S,M,MDUM /PLAN/CR
      PI=3.141593
      CONST=PI*FLOAT(2*M+1)/2.
      XEDGE = CR+Y*(1.-CR)/S
      X2Y2 = SQRT(XEDGE**2+Y*Y)
      XEDGE=X2Y2/SQRT(1.+S*S)
      GDELT=-CONST*COS(CONST*XEDGE)
      XGWT =-GDELT*((Y-YPT)*XI(Y,YPT)+(Y+YPT)*XI(Y,-YPT))/(4.*PI)
      RETURN
      END
```

```
                  FUNCTION XI(Y,YPT)
C
C XI GIVES CONTRIBUTION FROM WAKE VORTICITY
C              ARROW WING CONFIGURATION
C
C         ARGUMENT LIST
C              Y:    SPANWISE COORDINATE; NON-D BY MAXIMUM LENGTH
C              YPT:  VORTEX SPANWISE LOCATION
C
      COMMON XPT,YDUM,S,N,M /PLAN/CR /SEC/ZPT
      A=(Y-YPT)*(Y-YPT)+ZPT*ZPT
      XEDGE = CR+Y*(1.-CR)/S
       B = XEDGE -XPT
      XI=(1.-B/SQRT(A+B*B))/A
      RETURN
      END
```

```
                    SUBROUTINE CHDWS
C
C  CHDWS: EVALUATION OF CHORDWISE INTEGRAL USING LEGENDRE-GAUSS
C        QUADRATURE FOR CONTRIBUTION TO VELOCITY ON VORTEX
C
      DIMENSION BETA(10),THETA(10),GX(10),AL(5)
       COMMON /JACOB/ DKDY(10),DKDZ(10),DKVDY(10),XDWDY(5),XDWDZ(5),
     C XDVDY(5) /GAUN/GN(10,10),WN(10,10) /MODES/NOCM,NCSM
     C /WOV2/AR(4,5,5),ALS(5,5),NINC,CR(5),TKR(10),XOC,SOS,Y,Z,YMN,ZMZ,
     C RSOR,ETA,GAUSX(10),PO2,NCP,MP,N,X,GZ,J1,J2,GS,YMN2,ZMZ2,CSR
      COMMON /SOWSH/ TKVR(10),AVR(4,5,5),CVR(5)
C
C INITIALIZE SUMMATION VARIABLES
       DO 1  I=1,NOCM
       CR(I)=0.0
      CVR(I)=0.0
      XDWDY(I)=0.0
      XDWDZ(I)=0.0
      XDVDY(I)=0.0
 1     CONTINUE
C
C CALCULATE LOCATION OF LEADING EDGE AND LOCAL SEMICHORD
C      NON-D BY ROOT SEMICHORD
       ELE=XLE(N,GS)
      SEMICD=B(N,GS)
C
C    IF(RSOR-.1)>0, THE INTEGRAL IS EVALUATED AS A SINGLE INTEGRAL
      IF (RSOR-0.1) 21,3,3
   3  IF (NINC) 4,4,7
 4     NINC=2
       DO 5 I=1,NCP
C
C CALCULATE ANGULAR SPACING FOR INTEGRAL
       BETA(I)=(1.-GN(I,NCP))*PO2
      GX(I)=-COS(BETA(I))
      DO 5 J=1,NOCM


      ALS(J,I)=SIN(BETA(I)*FLOAT(J))/FLOAT(2**(2*J))*4.
C
C  ALS(J,I) = LOADING FUNCTIONS. REF: ASHLEY AND LANDAHL
 5       CONTINUE
  7    DO 6 I=1,NCP
 6       GAUSX(I)=X-(ELE+SEMICD*(1.+GX(I)) )
C
C  GAUSX = X-X1: NON-D BY ROOT SEMICHORD
C
C CALCULATE KERNELS FOR SURFACE INTEGRALS
       CALL KERNL
       WGHT=PO2
       DO 20 I=1,NCP
      CW=WN(I,NCP)*SIN(BETA(I))*WGHT
      DO 20 J=1,NOCM
      CR(J)=ALS(J,I)*CW*TKR(I)+CR(J)
      CVR(J)=ALS(J,I)*CW*TKVR(I)+CVR(J)
      XDWDY(J)=XDWDY(J)+ DKDY(I)*ALS(J,I)*CW
      XDWDZ(J)=XDWDZ(J)+ DKDZ(I)*ALS(J,I)*CW
      XDVDY(J)=XDVDY(J)+DKVDY(I)*ALS(J,I)*CW
 20     CONTINUE
       GO TO 50
C
C   FOR RSOR-.1<0, THE CHORDWISE INTEGRAL IS COMPUTED BY 2 LEGENDRE-
C   GAUSS QUADRATURES TO HANDLE FINITE JUMP IN KERNEL AT X-X1=Y-Y1=0
C
C IF X IS OFF WING AT Y, USE SINGLE INTEGRAL
 21   IF(X-ELE) 3,3,220
 220   IF(X-(ELE+2.*SEMICD)) 22,3,3
  22  THBD=ARCOS((ELE+SEMICD-X)/SEMICD)
       K=-1
       WGHT=THBD/2.
       DO 23 I=1,NCP
       THETA(I)=(1.-GN(I,NCP))*THBD/2.
 23        GAUSX(I)=X-(ELE+SEMICD*(1.-COS(THETA(I))))
       GO TO 35
```

```
24        WGHT=PO2-WGHT
      K=1
      DO 25 I=1,NCP
      THETA(I)=THBD+(1.-GN(I,NCP))*(PO2-THBD/2.)
  25  GAUSX(I)=X-(ELE+SEMICD*(1.0-COS(THETA(I))))
C
C   GAUSX = X-X1
C
C CALCULATE KERNELS FOR SURFACE INTEGRALS
 35       CALL KERNI
C
C DO CHORDWISE INTEGRALS
      DO 40  I=1,NCP
     CW=WN(I,NCP)*SIN(THETA(I))*WGHT
     DO 40 J=1,NCCM
     AL(J)=SIN(THETA(I)*FLOAT(J))/FLOAT(2**(2*J)) *4.
     CR(J)=AL(J)*CW*TKR(I)+CR(J)
     CVR(J)=AL(J)*CW*TKVR(I)+CVR(J)
     XDWDY(J)=XDWDY(J)+ DKDY(I)*AL(J)*CW
     XDWDZ(J)=XDWDZ(J)+ DKDZ(I)*AL(J)*CW
     XDVDY(J)=XDVDY(J)+DKVDY(I)*AL(J)*CW
C
C   CR,CVR,XDWDY,XDWDZ,XDVDY  ARE THE CHORDWISE INTEGRALS
 40       CONTINUE
C
C LOOP FOR SECOND INTEGRAL
      IF(K) 24,50,50
  50 CONTINUE
C
C        PRIMARY OUTPUT  CR,CVR,XDWDY,XDWDZ,XDVDY ARE RETURNED THROUGH
C             COMMON BLOCK TO CALLING PROGRAM
C
 60        RETURN
      END
```

```
                SUBROUTINE COLPT(NCORD,NSPAN,XPT,YPT)
C
C  COLPT CALCULATES COLLOCATION POINTS ON PLANFORM
C
C        ARGUMENT LIST
C             NCORD: NO. OF CHORDWISE POINTS
C             NSPAN: NO. OF SPANWISE POINTS
C             XPT: CHORDWISE POINT; NORMALIZED BY 1
C             YPT: SPANWISE POINT; NORMALIZED BY 1
C
      DIMENSION XPT(5),YPT(5)
      PI = 3.141593
      DO 10 I=1,NCORD
   10 XPT(I) = COS(PI*FLOAT(2*I-1)/FLOAT(4*NCORD))
      DO 20 J=1,NSPAN
   20 YPT(J) = COS(FLOAT(J)*PI/FLOAT(2*NSPAN+1))
      RETURN
      END
```

```
      SUBROUTINE DFNCT(GF,M,X,DFN)
C
C DFNCTN CALCULATES DERIVATIVES OF CHEBYSHEV POLYNOMIALS
C
C        ARGUMENT LIST
C              GF:  COEFFICIENTS OF POLYNOMIAL APPROXIMATION
C              M:   ORDER OF POLYNOMIAL APPROXIMATION
C              X:   CHORDWISE ARGUMENT
C              DFN: VALUE OF DERIVATIVE OF FUNCTION BEING APPROXIMATED
C
      DIMENSION GF(5),CHEBY2(5),DCHEBY(5)
       CHEBY2(1)=1.
       CHEBY2(2)=4.*X*X-1.
      IF(M.LT.3) GO TO 80
      CSQ=CHEBY2(2)-1.
      DO 60 L=3,M
      CHEBY2(L)=CSQ*CHEBY2(L-1)-CHEBY2(L-2)
   60 CONTINUE
   80 CONTINUE
      DO 100 I=1,M
      DCHEBY(I)=CHEBY2(I)*FLOAT(2*I-1)
  100 CONTINUE
      DFN=0.0
      DO 500 I=1,M
  500 DFN=DFN+GF(I)*DCHEBY(I)
      RETURN
      END
```

```
      SUBROUTINE DGWGM(DGWGM2,DGWGM4,GWGMW1)
C
C  DGWGM CALCULATES CONTRIBUTION TO W FROM LEADING-EDGE VORTICES
C
C        ARGUMENT LIST
C              DGWGM2:  CONTRIBUTION TO Y DERIVATIVE
C              DGWGM4:  CONTRIBUTION TO Z DERIVATIVE
C              GWGMW1:  CONTRIBUTION TO W VELOCITY
C
      DIMENSION TCHEBY(5),UCHEBY(5),SUM2(5,5),SUM4(5,5),A(4),B(4),
     C DGWGM2(5,5),DGWGM4(5,5),GWGMW1(5),SUM(5),DFWDY(5),DFWDZ(5)
      COMMON XPI,YPI,S,MDUM,MPDUM /GAUS/G(24),H(24)
      COMMON/ GVOR/GYVOR(5),GZVOR(5) /VLOC/LMAX/MODES/NOCM,NOSM
C
      PI= 3.141593
      CONST2= 1./(8.*PI)
C
C  CONST2 = 1/(4.*PI) * 1/2 TO SCALE INTEGRAL
C
C INITIALIZE SUMMATION VARIABLES
      DO 100 I=1,5
      SUM(I)=0.
      DO 100 J=1,5
      SUM2(I,J) = 0.
      SUM4(I,J) = 0.
  100 CONTINUE
C
C WANT FOUR CALLS TO INTEGRATION ROUTINE
      DATA A,B/0.,.125,.25,.5,.125,.25,.5,1./
      DO 300 IC=1,4
      BMA = B(IC)-A(IC)
      BPA=B(IC)+ A(IC)
C
C DO CHORDWISE INTEGRALS BY 24-POINT GAUSS. QUAD.
      DO 200 J=1,24
C
```

```
C CALCULATE INTERMEDIATE FACTORS
      X = (RMA*G(J)+RPA)/2.
C
C CALCULATE CHEBYSHEV POLYNOMIALS
      CALL TUCHEB(LMAX,X,TCHEBY,UCHEBY)
C
C CALCULATE VORTEX POSITION AND DERIVATIVES
       CALL FNCTN(GYVOR,LMAX,X,YVORT)
      CALL FNCTN(GZVOR,LMAX,X,ZVORT)
       CALL DFNCT(GYVOR,LMAX,X,DYVORT)
C
C CALCULATE INTERMEDIATE FACTORS
      XDIFF= X-XPT
      YDIFF= YVORT-YPT
      YSUM = YVORT+YPT
       TERM1= XDIFF*XDIFF+YDIFF*YDIFF+ZVORT*ZVORT
       TERM2= XDIFF*XDIFF+ YSUM*YSUM+ ZVORT*ZVORT
       TERM3=TERM1*SQRT(TERM1)
      TERM4= TERM2*SQRT(TERM2)
C
C CONTRIBUTION TO DOWNWASH FROM LEADING-EDGE VORTEX
      FW = (XDIFF*DYVORT-YDIFF)/TERM3 + (XDIFF*DYVORT-YSUM)/TERM4
      DO 140 L=1,LMAX
C
C CHANGE IN DOWNWASH CONTRIBUTION DUE TO CHANGE IN VORTEX POSITION
       DFWDY(L)=(XDIFF*FLOAT(2*L-1)*UCHEBY(L)-TCHEBY(L)
     C -3.*(XDIFF*DYVORT-YDIFF)*YDIFF*TCHEBY(L)/TERM1)/TERM3
     C +(XDIFF*FLOAT(2*L-1) * UCHEBY(L)-TCHEBY(L)
     C -3.*(XDIFF*DYVORT-YSUM)*YSUM*TCHEBY(L)/TERM2)/TERM4
      DFWDZ(L)=    ZVORT*   TCHEBY(L)*((XDIFF*DYVORT-YDIFF)/
     C (TERM1*TERM3) + (XDIFF*DYVORT-YSUM)/(TERM2*TERM4))
  140 CONTINUE
      DO 150 MO=1,NOCM
      M=MO-1
      CONST = FLOAT(2*M+1)/2.*PI
      GGAM= SIN(CONST*X)



C
C  GGAM = LEADING-EDGE VORTEX STRENGTH
      SUM(MO) = GGAM*FW*W(J) +SUM(MO)
      DO 150 L=1,LMAX
      SUM2(MO,L) = GGAM*DFWDY(L)*W(J) +SUM2(MO,L)
      SUM4(MO,L) = GGAM*DFWDZ(L)*W(J) +SUM4(MO,L)
  150 CONTINUE
  200 CONTINUE
C DETERMINE WEIGHTING FACTOR
      CONST=1.
      IF(IC.GT.1) CONST=.5
       DO 400 MO=1,NOCM
          SUM(MO) = CONST*SUM(MO)
      DO 400 L=1,LMAX
         SUM2(MO,L) = CONST*SUM2(MO,L)
         SUM4(MO,L) = CONST*SUM4(MO,L)
  400 CONTINUE
  300 CONTINUE
      DO 500 MO=1,NOCM
       GWGMW1(MO) = CONST2*SUM(MO)
      DO 500 L=1,LMAX
       DGWGM2(MO,L) = CONST2*SUM2(MO,L)
      DGWGM4(MO,L) = -3.*       CONST2*SUM4(MO,L)
  500 CONTINUE
C
C         PRIMARY OUTPUTS  GWGMW1,DGWGM2,DGWGM4 PASSED THROUGH
C                ARGUMENT LIST TO CALLING PROGRAM
      RETURN
      END
```

```
      SUBROUTINE DGWV(DGWTY,DGVTY,DGWTZ,DGVTZ)
C
C  DGWV PROVIDES CONTRIBUTION TO JACOBIAN FROM WAKE
C
C              ARGUMENT LIST
C              DGWTY: CHANGE IN GWT FROM CHANGE IN YV
C              DGVTY: CHANGE IN GVT FROM CHANGE IN YV
C              DGWTZ: CHANGE IN GWT FROM CHANGE IN ZV
C              DGVTZ: CHANGE IN GVT FROM CHANGE IN ZV
C
C FACTOR OF  I(2*L-1) OUTSIDE OF SUBROUTINE
C
      COMMON XPI,YVORT,S,MDUM,MPDUM/SEC/ZVORT/GAUS/G(24),W(24)
     C /MODES/NOCM,NOSM /PLAN/CR
      DIMENSION SUM(5,4),DGWTY(5),DGVTY(5),DGWTZ(5),DGVTZ(5)
C
      PI=3.14159
C
C INITIALIZE SUMMATION VARIABLES
      DATA SUM/20*0.0/
C
C DO SPANWISE INTEGRAL FROM 0. TO S
      DO 200 J=1,24
C
C CALCULATE ABSCISSAS FOR GAUSSIAN QUADRATURE
      Y=S*(1.+G(J))/2.
C
C CALCULATE INTERMEDIATE FACTORS
      DIPYDY=DIDY(Y,YVORT)
      DIMYDY=-DIDY(Y,-YVORT)
      DIPDZ=DIDZ(Y,YVORT)
      DIMDZ=DIDZ(Y,-YVORT)
      YDIFF=Y-YVORT
      YSUM=Y+YVORT
      XEDGE = CR+Y*(1.-CR)/S
       X2Y2 = SQRT(XEDGE**2+Y*Y)
      XEDGE=X2Y2/SQRT(1.+S*S)
      XIP=XI(Y,YVORT)
      XIM=XI(Y,-YVORT)
C
C DO FOR ALL MODES
      DO 100 MO=1,NOCM
      M=MO-1
      CONST=PI*FLOAT(2*M+1)/2.
      GDELT=-CONST*COS(CONST*XEDGE)
      YDGWTY=GDELT*(XIP-XIM-YDIFF*DIPYDY-YSUM*DIMYDY)
       YDGVTY=GDELT*(DIPYDY-DIMYDY)
      YDGWTZ=GDELT*(YDIFF*DIPDZ+YSUM*DIMDZ)
      YDGVTZ=GDELT*(ZVORT*(DIPDZ-DIMDZ)+XIP-XIM)
      SUM(MO,1)=SUM(MO,1)+YDGWTY*W(J)
      SUM(MO,2)=SUM(MO,2)+YDGVTY*W(J)
      SUM(MO,3)=SUM(MO,3)+YDGWTZ*W(J)
      SUM(MO,4)=SUM(MO,4)+YDGVTZ*W(J)
  100 CONTINUE
  200 CONTINUE
C
C  CONST FROM S/2. * 1/(4.*PI)
      CONST=S/(8.*PI)
       DO 300 MO=1,NOCM
      DGWTY(MO)=CONST*SUM(MO,1)
      DGVTY(MO)=-ZVORT*CONST*SUM(MO,2)
      DGWTZ(MO)=-CONST*SUM(MO,3)
       DGVTZ(MO)=-CONST*SUM(MO,4)
      DO 300 K=1,4
       SUM(MO,K)=0.0
  300 CONTINUE
C
C  RESULTS PASSED TO GVCTR THROUGH ARGUMENT LIST
      RETURN
      END
```

```
      SUBROUTINE FNCTN(CF,M,X,FN)
C
C FNCTN EVALUATES CHEBYSHEV POLYNOMIALS
C
C        ARGUMENT LIST
C              CF:  COEFFICIENTS OF POLYNOMIAL APPROXIMATION
C              M:   ORDER OF POLYNOMIAL APPROXIMATION
C              X:   CHORDWISE POINT OF INTEREST
C              FN:  VALUE OF FUNCTION
C
      DIMENSION CF(5),CHEBY(5)
C USE CHEBYSHEV POLYNOMIALS OF THE FIRST KIND
      CHEBY(1)=X
      CSQ=4.*X*X-2.
      CHEBY(2)=(CSQ-1.)*X
      IF(M.LT.3) GO TO 80
      DO 60 L=3,M
      CHEBY(L)=CSQ*CHEBY(L-1)-CHEBY(L-2)
   60 CONTINUE
C CALCULATE FUNCTION FROM POLYNOMIAL CONTRIBUTIONS
   80 FN=0.0
      DO 500 I=1,M
  500 FN=FN+CF(I)*CHEBY(I)
      RETURN
      END
```

```
      SUBROUTINE FUNCTN(NOSM,S,F)
C
C   FUNCTN: SPANWISE LOADING FUNCTIONS
C
C       ARGUMENT LIST
C            NOSM:  NO. OF SPANWISE HORSESHOE VORTEX MODES
C            S:     SPANWISE COORDINATE; NON-D BY SEMISPAN
C            F:     VALUE OF FUNCTION
C
      DIMENSION F(5)
C
C USE CHEBYSHEV POLYNOMIALS AS LOADING FUNCTIONS
      SQ=S*S
      R=SQRT(1.0-SQ)
      F(1)=R
      F(2)=R*(4.*SQ-1.)
      C=4.*SQ-2.
      DO 20 J=3,NOSM
   20 F(J)=C*F(J-1)-F(J-2)
      RETURN
      END
```

```
                 SUBROUTINE GAUS1D(C,D,ENTGL,F)
C
C  GAUS1D PERFORMS 1-D INTEGRATION BY 24-POINT GAUSSIAN QUADRATURE
C
C        ARGUMENT LIST
C             C:    LOWER LIMIT OF INTEGRAL
C             D:    UPPER LIMIT OF INTEGRAL
C             ENTGL: VALUE OF INTEGRAL
C             F:    FUNCTION TO BE INTEGRATED
C
       COMMON /GAUS/G(24),W(24)
C
C INITIALIZE SUMMATION VARIABLES
      SUM=0.0
      DC=D-C
      DAC=D+C
      DO 200 J=1,24
      XNEW=(DC*G(J)+DAC)/2
      SUM = SUM+F(XNEW)*W(J)
  200 CONTINUE
      ENTGL = DC*SUM/2.
      RETURN
      END
```

```
                 SUBROUTINE GVCTR(NOCP)
C
C  GVCTR CALCULATES FORCE ON VORTEX AND CORRESPONDING DERIVATIVES
C
C        ARGUMENT LIST
C             NOCP:  NO. OF COLLOCATION POINTS
C
      COMMON/MODES/NOCM,NOSM/VLOC/LMAX/GVOP/GYVOR(5),GZVOR(5)
     C /GVEC/  A(5,5),GC(5),N1,FSUBY(5),FSUBZ(5),PI,SINALF,NOFP
     C/VORT/YVOR,ZVOR/ /XPI,YVORT,S,M,MP/ SEC/ZVORT
     C                  /YACGB/XACGB(35,35),SAWH(5,5),SAVW(5,5),
     C DAWDY(5,5),DAWDZ(5,5),DAVDY(5,5),DAVDZ(5,5)
      DIMENSION SGVV(5),SGWV(5),DGWDY(5),DGWDZ(5),
     CDGVDY(5),DGVDZ(5),DGWTY(5),DGVTY(5),DGWTZ(5),DGVTZ(5),TCHEBY(5),
     CUCHEBY(5),DGVGM2(5,5),DGWGM2(5,5),DGVGM4(5,5),DGWGM4(5,5)
      EXTERNAL        XGWT, XGVT,XGWL,XGVL
C
      N1=N1+1
      NMODT= NOCM*NOSM+NOCM
C
C  NMODT = TOTAL NO. OF VORTICITY MODES
C
C CALCULATE VORTEX LOCATION AND DERIVATIVES
      CALL FNCTN(GYVOR,LMAX,XPI,YVORT)
      CALL FNCTN(GZVOR,LMAX,XPI,ZVORT)
      CALL DFNCT(GYVOR,LMAX,XPI,DYVORT)
      CALL DFNCT(GZVOR,LMAX,XPI,DZVORT)
C
C OUTPUT CONTROL POINT LOCATION
      WRITE(6,910)   XPI,YVORT,ZVORT
C
C CALCULATE LEADING-EDGE VORTEX STRENGTH AND DERIVATIV'
C        CALCULATE GAMMA AND DGAMMA/DX
      DGAMM = 0.
      GAMMA=0.0
      DO 600 MO=1,NOCM
```

```
      M=MQ-1
      CONST=FLOAT(2*M+1)/2.*PI
      GAMMA=GAMMA+GO(MQ)*SIN(CONST*XPI)
  600 DGAMM=DGAMM+GO(MQ)*CONST*COS(CONST*XPI)
C
C INITIALIZE SUMMATION VARIABLES
      W1=0.0
      V1=0.0
C
C CALCULATE V1,W1 AT XPI,YVORT,ZVORT
C
C CONTRIBUTION OF LEFT HAND VORTEX AND WAKE
C CALCULATE INTERMEDIATE FACTORS
      XDIFF=1.-XPI
      YSUM=YVOR      +YVORT
      ZDIFF=ZVOR      -ZVORT
      TERM2=YSUM*YSUM+ZDIFF*ZDIFF
      TERM4=TERM2+XDIFF*XDIFF
      ROOT4=SQRT(TERM4)
C
C CALCULATE CONTRIBUTION AFT OF X = 1.
      GWGML2=-YSUM/TERM2*(1.-XDIFF/      ROOT4               )/(4.*PI)
      GVGML2=GWGML2*ZDIFF/YSUM
C
C CONTRIBUTION FROM WING
      CALL GWVD(SGVV,SGWV,DGVDY,DGVDZ,DGWDY,DGWDZ)
C
C CONTRIBUTION FROM WAKE AND LEADING-EDGE VORTEX FORWARD OF X = 1.
      DO 450 MQ=1,NOCM
      M=MQ-1
      N2= NOCM*NCSM+MQ
      CALL GAUSID(   0.0,S,GWT,XGWT)
      CALL GAUSID(   0.0,S,GVT,XGVT)
      CALL GAUSID(   0.0,1.0,GW,XGWL)
      CALL GAUSID(   0.0,1.0,GV,XGVL)
      CONSTG = FLOAT(2*M+1)/2.*PI



C
C SUM CONTRIBUTIONS TO VELOCITY COEFFICIENTS
      GWV  = SGWV(MQ) + GW*GWGML2+GWT
      GVV  = SGVV(MQ) + GV*GVGML2+GVT
      GWGML2=-GWGML2
      GVGML2=-GVGML2
      TERMG = (DGAMM*SIN(CONSTG*XPI)/GAMMA    -CONSTG*COS(CONSTG*XPI))
     C /GAMMA
C
C CALCULATE DERIVATIVES W.R.T. VORTICITY COEFFICIENTS  GO
      XACOB(N1,N2) = GWV  + ZVORT*TERMG
      XACOB(N1+NCFP,N2)  = -GVV  -(YVORT-S*XPI)*TERMG
C
C CALCULATE VELOCITY COMPONENTS
      W1=GO(MQ)*GWV+W1
      V1=GO(MQ)*GVV+V1
       DO 450 MPP=1,NOSM
      AWV=SAWW(MQ,MPP)
      AVV=SAVW(MQ,MPP)
      N2 = MQ+(MPP-1) *NOCM
C
C CALCULATE DERIVATIVES W.R.T. HORSESHOE VORTICITY COEFFICIENTS, A
      XACOB(N1,N2) = AWV
      XACOB(N1+NCFP,N2)  = -AVV
C
C CALCULATE VELOCITY AT VORTEX
      W1=W1+A(MQ,MPP)*AWV
      V1=V1+A(MQ,MPP)*AVV
  450 CONTINUE
C
C CALCULATE FORCE COMPONENTS IN Y AND Z DIRECTIONS
      FY=-   (       (DZVORT   -W1-SINALF)+DGAMM*ZVORT/GAMMA)
      FZ=            (DYVORT   -V1)+DGAMM*(YVORT-S*XPI)/GAMMA
C
C OUTPUT FORCES
      WRITE(6,930) GAMMA,V1,W1,FY,FZ
```

```
  221 FSUBY(NI-NCCP ) = FY
      FSUBZ(NI-NCCP ) = FZ
C
C CALCULATE DERIVATIVES W.R.T. VORTEX POSITION COEFFICIENTS
C
C CALCULATE CHEBYSHEV POLYNOMIALS
      CALL TUCHEB (LMAX,XPI,TCHEBY,UCHEBY)
C
C CONTRIBUTION FROM LEADING-EDGE VORTEX
      CALL VORINT(DGWGM2,DGVGM2,DGWGM4,DGVGM4)
C
C CONTRIBUTION FROM WAKE
      CALL DGWV(DGWTY,DGVTY,DGWTZ,DGVTZ)
      TERM5=TERM4*ROOT4
      TERM6=1.-XDIFF/ROOT4
C
C CONTRIBUTION FROM VORTEX AFT OF X =1.
      DGWG1=1./TERM2*(   TERM6          *(-1.+2.*YSUM*YSUM/TERM2)
     C-YSUM*YSUM*XDIFF/  TERM5       )/(4.*PI)
      DGVG1=ZDIFF*YSUM/TERM2*(2./TERM2*  TERM6
     C-XDIFF/    TERM5     )/(4.*PI)
      DGWG3=DGVG.
      DGVG3=1./TERM2*(    TERM6          *(-1.+2.*ZDIFF*ZDIFF/TERM2)
     C-ZDIFF*XDIFF*ZDIFF/TERM5    )/(4.*PI)
C
C CONTRIBUTION FOR ALL MODES
      DO 220 LDUM=1,LMAX
      L=LDUM
C
C INITIALIZE SUMMATION VARIABLES
       DW1DGY=0.0
       DV1DGY=0.0
       DW1DGZ=0.0
       DV1DGZ=0.0
C
C CALCULATE CONTRIBUTION FROM LEADING VORTICES AFT OF WING


      DGWGM1=DGWG1*(1.+TCHEBY(L))
      DGVGM1=DGVG1*(1.+TCHEBY(L))
      DGWGM3=DGWG3*(1.-TCHEBY(L))
      DGVGM3=DGVG3*(1.-TCHEBY(L))
       DO 310 MO=1,NOCM
C
C CALCULATE CONTRIBUTION FROM WAKE
      DGWTDY=TCHEBY(L)*DGWTY(MO)
      DGVTDY=TCHEBY(L)*DGVTY(MO)
      DGWTDZ=DGWTZ(MO)*TCHEBY(L)
      DGVTDZ=DGVTZ(MO)*TCHEBY(L)
C
C CALCULATE CONTRIBUTION FROM LEADING-EDGE VORTEX
      DGWGMY=DGWGM1+DGWGM2(MO,L)
      DGVGMY=DGVGM1+DGVGM2(MO,L)
      DGWGMZ=DGWGM3+DGWGM4(MO,L)
      DGVGMZ=DGVGM3+DGVGM4(MO,L)
C
C CALCULATE CONTRIBUTION FROM WING
      DSGWDY=DGWDY(MO)*TCHEBY(L)
      DSGVDY=DGVDY(MO)   EBY(L)
      DSGWDZ=DGWDZ(MO)   EBY(L)
      DSGVDZ=DGVDZ(MO)   BY(L)
C
C SUM CONTRIBUTIONS FOR    IVATIVE COEFFICIENTS
      DW1DGY=DW1DGY+GO(MO.*(DGWTDY+DGWGMY+DSGWDY)
      DV1DGY=DV1DGY+GO(MO)*(DGVTDY+DSGVDY     +DGVGMY)
      DW1DGZ=DW1DGZ+GO(MO)*(DGWTDZ+DGWGMZ+DSGWDZ)
      DV1DGZ=DV1DGZ+GO(MO)*(DGVTDZ+DSGVDZ+DGVGMZ)
C
      DGWGM1=-DGWGM1
      DGVGM1=-DGVGM1
      DGWGM3=-DGWGM3
      DGVGM3=-DGVGM3
C
C CONTRIBUTION FROM HORSESHOE VORTICITY
```

```
      DO 310 MPP=1,NOSM
      DWIDGY = DWIDGY + A(MQ,MPP)*DAWDY(MQ,MPP)*TCHEBY(L)
      DVIDGY = DVIDGY + A(MQ,MPP)*DAVDY(MQ,MPP)*TCHEBY(L)
      DWIDGZ = DWIDGZ + A(MQ,MPP)*DAWDZ(MQ,MPP)*TCHEBY(L)
      DVIDGZ = DVIDGZ + A(MQ,MPP)*DAVDZ(MQ,MPP)*TCHEBY(L)
  310 CONTINUE
C
C CALCULATE CONTRIBUTION TO JACOBIAN
C        CALCULATE DFY/DGYV   DFZ/DGYV   DFY/DGZV   DFZ/DGZV
      DFYDGY=          DWIDGY
      DFYDGZ=-(FLOAT(2*L-1)*UCHEBY(L)-DWIDGZ)-DGAMM*TCHEBY(L)/GAMMA
      DFZDGY=(FLOAT(2*L-1)*UCHEBY(L)-DVIDGY)+DGAMM*TCHEBY(L)/GAMMA
      DFZDGZ=-          DVIDGZ
      N2 = L+NMODT
      XACOB(N1,N2) = DFYDGY
      XACOB(N1+NCFP,N2) = DFZDGY
      N2= LMAX+L+NMODT
      XACOB(N1,N2) = DFYDGZ
      XACOB(N1+NCFP,N2) = DFZDGZ
  220 CONTINUE
C
C  OUTPUTS PASSED TO MAIN PROGRAM THROUGH COMMON STATEMENTS
C
  910 FORMAT(T20,' X =',F10.4,5X,'YV(X)=',F10.4,5X,'ZV(X)=',F10.4)
  930 FORMAT(' GAMMA=',E12.4,5X,'VI=',E12.4,5X,'WI=',E12.4,5X,'FY=',
     1E12.4,5X,'FZ=',E12.4)
      RETURN
      END
```

```
                    SUBROUTINE GWVD(SGVV,SGWV,DGVDY,DGVDZ,DGWDY,DGWDZ)
C
C CALCULATES SGVW,SGWW AND THEIR DERIVATIVES FOR PROGRAM V
C
C          ARGUMENT LIST
C                SGVV:  CONTRIBUTION TO V FROM WING VORTICITY
C                SGWV:  CONTRIBUTION TO W VROM WING VORTICITY
C                DGVDY: CHANGE IN GVV DUE TO CHANGE DGY
C                DGVDZ: CHANGE IN GVV DUE TO CHANGE DGZ
C                DGWDY: CHANGE IN GWV DUE TO CHANGE DGY
C                DGWDZ: CHANGE IN GWV DUE TO CHANGE DGZ
C
       COMMON XPT,YPT,S,MDUM,MPDUM /GAUS/G(24),W(24)  /PLAN/CR
      COMMON/SEC/ZPT /MODES/NOCM,NOSM
      DIMENSION SGVV(5),DGWDY(5),DGVDY(5),DGVDZ(5),DGWQZ(5),
     C SGWV(5),ENTGD(5,6),SUM(5,6)
C
      PI=3.141593
C
C INITIALIZE SUMMATION VARIABLES
      DO 10 I=1,5
      DO 10 J=1,6
      ENTGD(I,J) =0.
      SUM(I,J) =0.
   10 CONTINUE
C
C DIVIDE WING INTO TWO SECTION ABOUT XPLUS
      XPLUS = XPT+.02
      IF (XPLUS.GT..98) XPLUS = .98
      IF (XPLUS.LT.CR) GO TO 50
C
C ESTABLISH LIMITS FOR SPANWISE INTEGRATION
      CPRIM1 = S*(1.+XPLUS)/2.
      DPRIM1= S*(-1.+XPLUS)/2.
      CPRIM2 = S*(1.-XPLUS)/((1.-CR)*2.)
      DPRIM2 = S*(1.-2.*CR+XPLUS)/((1.-CR)*2.)
```

```
   50 CONTINUE
C
C DO SURFACE INTEGRAL IN TWO 24X24 GAUSS. QUADRATURES
      DO 600 ICOUNT = 1,2
C
C DO SPANWISE INTEGRAL
      DO 200 J=1,24
       IF (ICOUNT.EQ.2) GO TO 30
      IF (XPLUS.GE.CR) GO TO 25
   20 Y= S*XPLUS*G(J)
      B = XPLUS
      GO TO 27
   25 Y= CPRIM1*G(J) + DPRIM1
      B = B7(Y)
   27 A = A1(Y)
      GO TO 40
   30 IF(XPLUS.GE.CR) GO TO 35
       Y = S*G(J)
      GO TO 37
   35 Y = CPRIM2*G(J) +DPRIM2
   37 A = A5(Y)
      B = B5(Y)
   40 CONTINUE
      AP = B-A
      BP=B+A
C
C DO CHORDWISE INTEGRAL
      DO 100 I=1,24
      X=(AP*G(I)+BP)/2.
C
C CALCULATE INTERMEDIATE QUANTITIES
      XDIFF=X-XPT
      YDIFF=Y-YPT
      TERM1=YDIFF*YDIFF+ZPT*ZPT+XDIFF*XDIFF
      ROOT1=SQRT(TERM1)
      TERM2=TERM1*ROOT1


      TERM3     =(XDIFF*X+YDIFF*Y)                   /TERM1
C
C DO FOR ALL MODES
      DO 400 MQ=1,NOCM
      M=MQ-1
      GVORS=GVORT(M,X,Y,S)
      GDELT=-Y*GVORS
      XGVW=GDELT/TERM2
      XGWW=GVORS*TERM3
      ENTGD(MQ,4)=ENTGD(MQ,4)+XGWW      *W(I)/ROOT1
      ENTGD(MQ,1 )=ENTGD(MQ,1 )+XGVW      *W(I)
       XDGWDY=(GDELT+3.*XGWW*YDIFF)/TERM2
      ENTGD(MQ,5)=ENTGD(MQ,5)+XDGWDY*W(I)
      XDGVDY=XGVW*YDIFF/TERM1
      ENTGD(MQ,2)=ENTGD(MQ,2)+XDGVDY*W(I)
      XDGVDZ=XGVW*(1.-3.*ZPT*ZPT     /TERM1)
      ENTGD(MQ,3)=ENTGD(MQ,3)+XDGVDZ*W(I)
      ENTGD(MQ,6) = ENTGD(MQ,6) + W(I)*XGWW/(ROOT1*TERM1)
  400 CONTINUE
  100 CONTINUE
      DO 300 MQ=1,NOCM
      DO 300 MC=1,6
      SUM(MQ,MC )=SUM(MQ, MC)+ENTGD(MQ,MC )*W(J)*AP
      ENTGD(MQ,MC )=0.0
  300 CONTINUE
  200 CONTINUE
C
C SELECT PROPER MULTIPLYING FACTOR
      IF (ICOUNT.EQ.2) GO TO 130
      IF (XPLUS.GE.CR) GO TO 125
       CONST = XPLUS
      GO TO 140
  125 CONST = (1.+XPLUS)*(1.-CR)/(CR*(1.-XPLUS))
      GO TO 140
  130 CONST = S/(8.*PI)
      IF (XPLUS.LT.CR) GO TO 140
```

```
      CONST = CONST*CR*(1.-XPLUS)/(1.-CR)/2.
  140 CONTINUE
      DO 500 MO=1,NOCM
C
C CALCULATE CONTRIBUTION TO W,V VELOCITY AT VORTEX FROM WING VORTICITY
C       WHICH FEEDS LEADING-EDGE VORTICIES
      SGWV(MO)=CONST*SUM(MO,4)
      SUM(MO,4) = SGWV(MO)
      SUM(MO,1) = SUM(MO,1) *CONST
      SGVV(MO)=-ZPT*SUM(MO,1)
C
C CALCULATE DERIVATIVES
      DGWDY(MO)=CONST*SUM(MO,5)
      SUM(MO,5) = DGWDY(MO)
      SUM(MO,2) = SUM(MO,2)*CONST
       DGVDY(MO)=-3.*ZPT       *SUM(MO,2)
      SUM(MO,3) = SUM(MO,3)*CONST
      DGVDZ(MO)=-       SUM(MO,3)
      SUM(MO,6) = SUM(MO,6)*CONST
      DGWDZ(MO) = -3.*ZPT*SUM(MO,6)
  500 CONTINUE
  600 CONTINUE
C
C          PRIMARY OUTPUT   SGVV,SGWV,DGVDY,DGVDZ,DGWDY,DGWDZ
C               PASSED TO CALLING PROGRAM THROUGH ARGUMENT LIST
C
      RETURN
      END
```

```
                 SUBROUTINE KERNL
C
C  KERNL: EVALUATION OF KERNEL FUNCTIONS FROM STEADY,NON-PLANAR,
C       INCOMPRESSIBLE LIFTING SURFACE THEORY. REF: ASHLEY AND LANDAHL
C
      COMMON /JACOB/DKDY(10),DKDZ(10),DKVDY(10),XDWDY(5),XDWDZ(5),
     C XDVDY(5) /SDWSH/ TKVR(10),AVR(4,5,5),CVR(5)
     C /WDV2/AR(4,5,5),ALS(5,5),NINC,CR(5),TKR(10),XOC,SOS,Y,Z,YMN,ZMZ,
     C RSQR,ETA,GAUSX(10),PO2,NCP,MP,N,X,CZ,J1,J2,GS,YMN2,ZMZ2,CSR
C
    5 DO 10 I=1,NCP
C NON-D X-XI BY SEMISPAN
      XME=GAUSX(I)*CSR
      XME2=XME*XME
      R2=RSQR+XME2
      R=SQRT(R2)
      G=1.0+XME/R
      C=XME/(R2*R)
      D=4.*G/(RSQR*RSQR)
      E=2./RSQR+3./R2
      H=2./RSQR*G+C
       F=G-ZMZ2*H
C
C CALCULATE KERNELS FOR SIDEWASH,DOWNWASH, AND DERIVATIVE INTEGRALS
      TKVR(I)=-ZMZ*YMN*H
      TKR(I)= F
      DKDY(I)=YMN*(-2.*F/RSQR+ZMZ2*D+C*(-1.+E*ZMZ2))
      DKVDY(I)= (2.*YMN2/RSQR-1.)*ZMZ*H      +ZMZ*YMN2*(D+C*E)
   10 DKDZ(I)=ZMZ*(-2.*F/RSQR-2.*H+ZMZ2*D+C*(-1.+ZMZ2*E))
C
C          PRIMARY OUTPUT   TKVR,TKR,DKDY,DKVDY,DKDZ
C               PASSED THROUGH COMMON BLOCK TO CALLING PROGRAM
C
   15 RETURN
      END
```

```
      SUBROUTINE TUCHEB(LMAX,X,TCHEBY,UCHEBY)
C
C CALCULATES U(2*L-1)  T(2*L-1) FOR SUBROUTINE VORINT
C
C        ARGUMENT LIST
C            LMAX:  ORDER OF POLYNOMIAL APPROXIMATION
C            X:     CHORDWISE POINT OF INTEREST
C            TCHEBY:  CHEBYSHEV POLYNOMIAL OF FIRST KIND
C            UCHEBY:  CHEBYSHEV POLYNOMIAL OF SECOND KIND
C
      DIMENSION TCHEBY(5),UCHEBY(5)
      TCHEBY(1)=X
      UCHEBY(1)=1.
      CSQ=4.*X*X-2.
      UCHEBY(2)=CSQ+1.
      TCHEBY(2)=(CSQ-1.)*X
      IF(LMAX.LT.3) GO TO 80
      DO 60 L=3,LMAX
      TCHEBY(L)=CSQ*TCHEBY(L-1)-TCHEBY(L-2)
      UCHEBY(L)=CSQ*UCHEBY(L-1)-UCHEBY(L-2)
   60 CONTINUE
   80 RETURN
      END
```

```
      SUBROUTINE VORINT(DGWGM2,DGVGM2,DGWGM4,DGVGM4)
C
C  VORINT CALCULATES EFFECT OF VORTEX CONTRIBUTIONS DUE TO CHANGE IN
C       VORTEX LOCATION
C
C        ARGUMENT LIST
C            DGWGM2:  CHANGE IN W CONTRIBUTION DUE TO CHANGE IN GYV
C            DGVGM2:  CHANGE IN V CONTRIBUTION DUE TO CHANGE IN GYV
C            DGWGM4:  CHANGE IN W CONTRIBUTION DUE TO CHANGE IN GZV
C            DGVGM4:  CHANGE IN V CONTRIBUTION DUE TO CHANGE IN GZV
C
      COMMON XPI,YPI,S,MDUM,MPDUM/SEC/ZPT /GAUS/G(24),W(24)
     C       /GVOR/GYVOR(5),GZVOR(5)/MODES/NCCM,NOSM/VLOC/LMAX
      DIMENSION TCHEBY(5),UCHEBY(5),CHEBY1(5),SUM(5,5,4),
     C          DGWGM2(5,5),DGVGM2(5,5),DGWGM4(5,5),DGVGM4(5,5)
     C         ,DFWDY(5),DFVDY(5),DFWDZ(5),DFVDZ(5)
C
C INITIALIZE SUMMATION VARIABLES
      DATA SUM              /100*0.0/
      PI=3.141593
C
C ALL INTEGRALS FROM 0 TO 1 DONE IN 1 24X24 LOOP
       CALL TUCHEB (LMAX,XPI,CHEBY1,UCHEBY)
C
C DO CHORDWISE INTEGRAL FROM 0 TO 1
      DO 200 J=1,24
      X=(G(J)+1.)/2.
C
C CALCULATE ARGUMENTS
      CALL TUCHEB (LMAX,X,TCHEBY,UCHEBY)
C
C CALCULATE VORTEX LOCATION
      CALL FNCTN(GYVOR,LMAX,X,YVORT)
      CALL FNCTN(GZVOR,LMAX,X,ZVORT)
      CALL DFNCT(GYVOR,LMAX,X,DYVORT)
      CALL DFNCT(GZVOR,LMAX,X,DZVORT)
```

```
C
C CALCULATE INTERMEDIATE FACTORS
      XDIFF=X-XPT
      YSUM=YVORT+YPT
      ZDIFF=ZVORT-ZPT
      TERM1=XDIFF*XDIFF+YSUM*YSUM+ZDIFF*ZDIFF
      TERM2=TERM1*SQRT(TERM1)
      TERM3=TERM2*TERM1
      DO 140 L=1,LMAX
      CHEBS=TCHEBY(L)+CHEBY1(L)
      CHEBD=TCHEBY(L)-CHEBY1(L)
C
C CALCULATE INTEGRANDS
      DFWDY(L)=(XDIFF*FLOAT(2*L-1)*UCHEBY(L)-CHEBS-3.*(XDIFF*DYVORT-
     CYSUM)*YSUM*CHEBS/TERM1)/TERM2
      DFVDY(L)=(ZDIFF-XDIFF*DZVORT)*YSUM*CHEBS/TERM3
      DFWDZ(L)=(XDIFF*DYVORT-YSUM)*ZDIFF*CHEBD/TERM3
      DFVDZ(L)=(CHEBD-XDIFF*FLOAT(2*L-1)*UCHEBY(L)-3.*(ZDIFF-XDIFF
     C*DZVORT)*ZDIFF*CHEBD/TERM1)/TERM2
  140 CONTINUE
C
C DO FOR ALL MODES
      DO 150 MQ=1,NOCM
      M=MQ-1
      CONST=FLOAT(2*M+1)/2.*PI
      GGAM=SIN(CONST*X)
C
C  GGAM = LEADING-EDGE VORTEX STRENGTH
      DO 150 L=1,LMAX
      SUM(MQ,L,1)=SUM(MQ,L,1)+GGAM*DFWDY(L)*W(J)
      SUM(MQ,L,2)=SUM(MQ,L,2)+GGAM*DFVDY(L)*W(J)
      SUM(MQ,L,3)=SUM(MQ,L,3)+GGAM*DFWDZ(L)*W(J)
      SUM(MQ,L,4)=SUM(MQ,L,4)+GGAM*DFVDZ(L)*W(J)
  150 CONTINUE
  200 CONTINUE
      CONST=1./(8.*PI)


C
C MULTIPLY SUMMATIONS BY APPROPRIATE CONSTANTS
      DO 300 MQ=1,NOCM
      DO 300 L=1,LMAX
      DGWGM2(MQ,L)=SUM(MQ,L,1)*CONST
      DGVGM2(MQ,L)=3.*SUM(MQ,L,2)*CONST
      DGWGM4(MQ,L)=-3.*SUM(MQ,L,3)*CONST
      DGVGM4(MQ,L)=-SUM(MQ,L,4)*CONST
      DO 300 K=1,4
      SUM(MQ,L,K)=0.0
  300 CONTINUE
C
C         PRIMARY OUTPUTS  DGWGM2,DGVGM2,DGWGM4,DGVGM4
C               PASSED THROUGH ARGUMENT LIST TO CALLING PROGRAM
C
      RETURN
      END
```

```
                 SUBROUTINE WOV(L)
C
C  WOV CALCULATES VELOCITY INFLUENCE COEFFICIENTS ON VORTEX FROM
C        HORSESHOE VORTICITY
C
C          ARGUMENT LIST
C               L:    NO. OF FORCE POINT
C
      DIMENSION S(4),W(4),F(5),YDWDY(4,5,5),YDWDZ(4,5,5),YDVDY(4,5,5)
      COMMON /SDWSH/ TKVR(10),AVR(4,5,5),CVR(5)
      COMMON /JACCB/DKDY(10),DKDZ(10),DKVDY(10),XDWDY(5),
     C XDWDZ(5),XDVDY(5), /GAUN/GN(10,10),WN(10,10) /MODES/NOCM,NOSM
     C /WOV2/AR(4,5,5),ALS(5,5),NINC,CR(5),TKR(10),XOC,SOS,Y,Z,YMN,ZMZ,
     C RSQR,ETA,GAUSX(10),PO2,NCP,MP,N,X,GZ,J1,J2,GS,YMN2,ZMZ2,CSR
     C /WOW1/ JS,NI(4) /YACON/XACCR(35,35),SAWW(5,5),SAVW(5,5),
     C DAWDY(5,5),DAWDZ(5,5),DAVDY(5,5),DAVDZ(5,5) /PLAN/CXMAX
C
      DATA S(3),S(4),W(3)/ -1.,0.,1./
C
C  S =LEFT-HAND LIMIT OF INTERVAL
C  W =LENGTH OF INTERVAL
      PO2=1.570796
C
C TRANSFORM NON-D BY CHORD TO NON-D BY SEMICHORD
      X=-1.+2.*XCC
       Y=SOS
       ZMZ=Z
C
C  Y = VORTEX SPANWISE POSITION; NON-D BY SEMISPAN
C  ZMZ = VORTEX VERTICAL POSITION; NON-D BY SEMISPAN
C
C CHECK IF POINT IS CLOSE TO WING TIP
      IF(SOS+ETA.LT.1.) GO TO 30
      ETA=1.-SOS
C NO REGION 2
      NI(2)=0
   30 CONTINUE
C
C CHECK IF POINT IS CLOSE TO CENTER LINE
      IF(SOS-ETA.GT.0.0) GO TO 40
      ETA=SOS
C NO REGION 4
      NI(4)=0
   40 S(2)=SOS+ETA
C
C OUTPUT LOCATION OF CONTROL POINT AND OTHER INFO
      WRITE(6,910) L,XCC,SOS,ETA,NI(2),NI(4),X,Z
       W(2)=1.0-S(2)
       W(4)=SOS-ETA
      S(1)=SOS-ETA
      W(1)=2.*ETA
C
C INITIALIZE SUMMATION VARIABLES
       DO 41  I=1,JS
       DO 41 N1=1,NOCM
       DO 41 N2=1,NOSM
       AR(I,N1,N2)=0.0
      AVR(I,N1,N2)=0.0
      YDWDY(I,N1,N2)=0.0
      YDWDZ(I,N1,N2)=0.0
      YDVDY(I,N1,N2)=0.0
   41  CONTINUE
C
C DO INTEGRALS OVER FOUR SPANWISE REGIONS
       DO 500 I=1,JS
  411  NSIP=NI(I)
C
C  NSIP =NO. OF INTEGRAL POINTS
      IF(NSIP.EQ.0) GO TO 500
C
C DO SPANWISE INTEGRAL
       DO 50  J=1,NSIP
```

```
      GS=S(I)-(GN(J,NSIP)-1.0)/2.*W(I)
C
C  GN(J,NSIP) =JTH ABSCISSA OF LEGENDRE-GAUSS QUADRATURE OF ORDER NSIP
      GY=GS
       YMN=Y-GY
      YMN2=YMN*YMN
      ZMZ2=ZMZ*ZMZ
      RSQR=YMN2+ZMZ2
      WT=WN(J,NSIP)*       W(I)/(2.*RSQR)
C
C  WN(J,NSIP) =JTH WTG. FUNCTION OF LEGENDRE-GAUSS QUADRATURE
C
C CALCULATE SPANWISE VORTICITY MODE
      CALL FUNCTN(NOSM,GS,F)
C
C DO CHORDWISE INTEGRAL
       CALL CHDWS
       DO 45 M=1,NOCM
       DO 45 NSF=1,NOSM
       AR(I,M,NSF)=AR(I,M,NSF)+CR(M)*F(NSF)*WT
      AVR(I,M,NSF)=AVR(I,M,NSF)+CVR(M)*F(NSF)*WT
      YDWDY(I,M,NSF)=YDWDY(I,M,NSF)+XDWDY(M)*F(NSF)*WT
      YDWDZ(I,M,NSF)=YDWDZ(I,M,NSF)+XDWDZ(M)*F(NSF)*WT
      YDVDY(I,M,NSF)=YDVDY(I,M,NSF)+XDVDY(M)*F(NSF)*WT
C
C  AR,AVR,ETC. ARE SURFACE INTEGRALS
  45   CONTINUE
  50   CONTINUE
 500    CONTINUE
C
C INITIALIZE SUMMATION VARIABLES
      DO 60 I=1,NCCM
      DO 60 J=1,NCSM
       SAWW(I,J) =0.
       SAVW(I,J) =0.
      DAWDY(I,J) =0.
      DAWDZ(I,J) =0.
      DAVDY(I,J) =0.
  60   CONTINUE
C
C SUM OVER ALL INTEGRATION REGIONS
       DO 70  I=1,NOCM
       DO 70  J=1,NCSM
       DO 65  MS=1,JS
      SAWW(I,J) = SAWW(I,J) + AR(MS,I,J)
      SAVW(I,J) = SAVW(I,J) + AVR(MS,I,J)
      DAWDZ(I,J) = DAWDZ(I,J) + YDWDZ(MS,I,J)*2.*CSR/CXMAX
      DAWDY(I,J) = DAWDY(I,J) +YDWDY(MS,I,J)*2.*CSR/CXMAX
      DAVDY(I,J) = DAVDY(I,J) + YDVDY(MS,I,J)*2.*CSR/CXMAX
   65 CONTINUE
      DAVDZ(I,J) = DAWDY(I,J)
  70  CONTINUE
C
C RESULTS ARE PASSED THROUGH COMMON STATEMENT
C
 910 FORMAT(1H0,'COLLOCATION PT.',I4,3X,'XOC=',F7.4,3X,'SOS=',F7.4,3X,
    C'ETA=',F7.4,3X,'NI(2)=',I3,3X,'NI(4)=',I3,3X,'X=',F7.4,3X,'Z=',
    CF7.4)
      RETURN
      END
```

```
                  SUBROUTINE WPDW(A,GO,NOCP,ALFA)
C
C WPDW  CALCULATES DOWNWASH AND PROVIDES CONTRIBUTION TO JACOBIAN
C
C         ARGUMENT LIST
C               A:   COEFFICIENTS FOR HORSESHOE VORTEX MODES
C               GO:  COEFFICIENTS FOR LEADING-EDGE VORTEX MODES
C               NOCP:  NO. OF COLLOCATION POINTS
C               ALFA:  ANGLE OF ATTACK (IN RADIANS)
C
      EXTERNAL         XGWT
      DIMENSION XPT(5),YPT(5),       A(5,5),GO(5)
     C ,GWGMW1(5),DGWGM2(5,5),DGWGM4(5,5),DWDGY(5),DWDGZ(5)
      COMMON XPI,YPJ,S,M,MP/GV   /GYVOR(5),GZVOR(5)/SEC/ZPT/PLAN/CR
     C /MODES/NOCM,NOSM /CONTR2/J3,J4 /VLOC/LMAX /YACOB/XACOB(35,35),
     C SAWW(5,5),SAVW(5,5),DAWDY(5,5),DAWDZ(5,5),DAVDY(5,5),DAVDZ(5,5)
      COMMON /CWPDW/ NCORD,NSPAN,COEFF(25,25),GWW(25,5),VR(25)
C
      PI=3.141593
      SINALF = SIN(ALFA)
C
C FOR POINTS ON WING, Z = 0.
      ZPT=0.
      NCORD2=NCORD+1
      IF(J3.EQ.1) NCORD2=NCORD
C
      NMOD=NOSM*NCCM
      NMODT=NMOD+NOCM
C
C  NMOD = NO. OF HORSESHOE VORTEX MODES
C  NMODT = TOTAL NO. OF VORTICITY MODES
C
C CALCULATE LOCATION OF COLLOCATION POINTS
      CALL COLPT(NCORD,NSPAN,  XPT,YPT)
      IF(J3.EQ.1) GO TO 300
      XPT(NCORD2)=(XPT(NCORD)+XPT(NCORD-1))/2.


  300 CONTINUE
C
C CALCULATE LOCATION OF VORTEX AT X = 1.
      CALL FNCTN(GYVOR,LMAX,1.,YVCR)
      CALL FNCTN(GZVOR,LMAX,1.,ZVCR)
C
C CALCULATE DOWNWASH RESIDUE AT COLLOCATION POINTS
      DO 400 I=1,NCORD2
      DO 400 J=1,NSPAN
C CALCULATE FIRST INDEX FOR MATRICES, N1
      N1=J+(I-1)*NSPAN
      IF(N1.GT.NOCP) GO TO 400
      XPI =((XLE(1,YPT(J))+1.)/2.   + B(1,YPT(J))*XPT(I))*CR
      YPJ=YPT(J)*S
C
C  CHORDWISE POINT; NON-D BY MAXIMUM LENGTH
C  YPJ: SPANWISE POINT; NON-D BY MAXIMUM LENGTH
C
C FORM INTERMEDIATE FACTORS
      YDIFF=YVCR     -YPJ
      YSUM=YVOR      +YPJ
      XDIFF=1.-XPI
      YDIFSQ=YDIFF*YDIFF
      YSUMSQ=YSUM*YSUM
      ZSQ=ZVOR      *ZVOR
      TERM1=YDIFSQ+ZSQ
      TERM2=YSUMSQ+ZSQ
      TERM3=TERM1+ XDIFF*XDIFF
      TERM4   TERM2+XDIFF*XDIFF
      TERM5 = SQRT(TERM3)
      TERM6 = SQRT(TERM4)
C
C CALCULATE CONTRIBUTIONS FROM LEADING-EDGE VORTICES AFT OF X = 1.
      GWGMW2=-(YDIFF/TERM1*(1.-XDIFF/TERM5)
     1+YSUM/TERM2*(1.-XDIFF/      TERM6              ))/(4.*PI)
       DGWGM1= - 1.*(((1.-2.*YDIFSQ/TERM1)*(1.-XDIFF/TERM5)
```

```
     C *YDIFSQ*XDIFF/(TERM1*TERM5))/TERM1*(1.-2.*YSUMSQ/TERM2)
     C *(1.-XDIFF/TERM6)   *YSUMSQ*XDIFF/(TERM4*TERM6))/TERM2)/(4.*PI)
      DGWGM1 = -2./VOR*(YDIFF/TERM1*(XDIFF/(TERM1*TERM5)
     C -2./TERM1*(1.-XDIFF/(       TERM5))) + YSUM/TERM2*(XDIFF/(TERM4
     C *TERM6) -2./TERM2*(1.-XDIFF/(       TERM6))))/(4.*PI)
C
C CALCULATE CONTRIBUTIONS FROM LEADING-EDGE VORTICES FORWARD OF X=1.
      CALL DGWGM(DGWGM2,DGWGM4,GWGMW1)
C
C INITIALIZE SUMMATION VARIABLES
       DO 320 L1=1,LMAX
      DWDGY(L1) = 0.
      DWDGZ(L1) = 0.
  320 CONTINUE
C
      DO 450 MQ=1,NOCM
      M=MQ-1
C
C CALCULATE CONTRIBUTIONS TO DOWNWASH COEFFICIENT FROM WAKE
      CALL GAUSID(   0.0,S,GWT,XGWT )
C
C CALCULATE DOWNWASH INFLUENCE COEFFICIENTS
      COEFF(N1,NMOD+MQ)=GWW(N1,MQ)+GWT+GWGMW2+GWGMW1(MQ)
      GWGMW2=-1.*GWGMW2
C
C CALCULATE DERIVATIVES FOR JACOBIAN
      DO 330 L1=1,LMAX
      DWDGY(L1) = DWDGY(L1) + GO(MQ)*(DGWGM2(MQ,L1)+ DGWGM1)
      DWDGZ(L1) = DWDGZ(L1) + GO(MQ)*(DGWGM4(MQ,L1) + DGWGM3)
  330 CONTINUE
      DGWGM1= -1.*DGWGM1
      DGWGM3= -1.*DGWGM3
  450 CONTINUE
      DO 430 L1=1,LMAX
      XACOB(N1,NMODT+L1) = DWDGY(L1)
      XACOB(N1,NMODT+LMAX+L1) = DWDGZ(L1)



  430 CONTINUE
  400 CONTINUE
      IF(J4.NE.1) GO TO 510
      DO 500 I=1,NOCP
C
C OUTPUT DOWNWASH INFLUENCE COEFFICIENTS , IF DESIRED
      WRITE(6,980) (COEFF(I,J),J=1,NMCOT)
  500 CONTINUE
  510 CONTINUE
C
C CALCULATE RESIDUE FROM DOWNWASH CONDITION
      DO 140 I=1,NOCP
      VR(I)= SINALF
      DO 140 J=1,NOCM
      VR(I) = VR(I)+ COEFF(I,NMOD+J)*GO(J)
      XACOB(I,NMOD+J)= COEFF(I,NMOD+J)
      DO 140 K=1,NOSM
      N2=J+NOCM*(K-1)
      VR(I) = VR(I)+ COEFF(I,N2)*A(J,K)
      XACOB(I,N2) = COEFF(I,N2)
  140 CONTINUE
C
C OUTPUT RESIDUE FROM DOWNWASH CONDITION
       WRITE(6,930)  (VR(I),I=1,NOCP)
C
C         PRIMARY OUTPUTS  VR,XACOB PASSED TO CALLING PROGRAM
C              THROUGH COMMON
C
  930 FORMAT(' DOWNWASH ON WING'/(5E14.5))
  940 FORMAT(' COLLOCATION POINT',I3,2F12.4,3X,'LOCAL X=', F12.4)
  980 FORMAT (10F13.5)
      RETURN
      END
```

SYMBOLS

a	vorticity coefficient
AR	aspect ratio
c	chord
C_N	normal force coefficient
C_p	pressure coefficient
F	force on right-hand vortex
i	unit vector in x-direction
j	unit vector in y-direction
K	kernel function for surface integral
k	unit vector in z-direction
l	dummy index
m	dummy index
n	dummy index
q	dummy index
r	radius vector from origin
S	surface of integration
s	semispan
t	thickness
U	free stream velocity
u	perturbation velocity in x-direction
v	perturbation in y-direction; vector v is total perturbation velocity
w	perturbation velocity in z-direction
x	chordwise coordinate
y	spanwise coordinate; with subscript v, represents leading-edge vortex spanwise location
z	vertical coordinate; with subscript v, represents leading-edge vortex vertical location

SYMBOLS (cont'd.)

α	angle of attack
Γ	leading-edge vortex strength
γ	spanwise vorticity component; vector γ is total vorticity
δ	chordwise vorticity component
η	spanwise coordinate nondimensionalized by semispan
θ	chordwise azimuthal coordinate
λ	leading-edge sweep angle
ω	complex plane

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